# Uniaxial Ferromagnets

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## Abstract

We discuss properties of uniaxially magnetic materials in the energetically optimal state. We work with the well-established micromagnetic Landau-Lifshitz model. Our goal is to better understand experimental observations using rigorous mathematical analysis of the model.

For our purposes, i.e. magnetostatics without applied field, a state is characterized by the magnetization $m$ of unit length in some sample in a domain $\Omega \subset \mathbb{R}^3$. As an auxiliary function we consider the magnetic field $h$ induced by the magnetization, also called the stray field. It is determined by the magnetization via Maxwell's equations to be the Helmholtz projection of the magnetization onto the space of curl-free vector fields. We consider a sample domain $\Omega = \mathbb{R}^2 \times (-t, t)$ of infinite extension in two directions and finite thickness $2t$ in the third. Our materials exhibit a crystalline anisotropy causing the energy to strongly favor the magnetization to point in the third direction called the easy axis.

The Landau-Lifshitz energy is the sum of three terms: The exchange energy penalizes spatial variation of $m$. The strength of the term is controlled by the exchange length $d$, a material parameter of typically a few hundred Ångström. The anisotropy term enforces a preference for the direction of the magnetization. The strength of the anisotropy is measured by a non-dimensional parameter called the quality factor $Q$. Finally, the magnetostatic or stray field energy is the integral of the squared strength of the field induced by the magnetization. Note that the stray field depends nonlocally on the magnetization. Thus the model is described by the sample thickness and the two material parameters, the exchange length and the quality factor.

In experiments with such materials, e.g. Kerr microscopy of neodymium-iron-boron magnets, the magnetization is observed to form patterns with domains and walls. Domains are regions of almost constant magnetization in direction of the easy axis. These domains are separated by walls, small, almost two-dimensional, areas in which the magnetization varies sharply.

Given these observations, the passage from the universal Landau-Lifshitz model to a reduced, sharp-interface model with magnetization indeed constant on domains and jumping at lower-dimensional sets has been heuristically justified in the physics literature. Inspired by these heuristics, mathematicians have been able to rigorously establish properties, notably about the scaling behavior of the energy, for the full model by using the intuition gained from analyzing sharp-interface models.

In this thesis we give a rigorous mathematical justification of the passage to a reduced model by establishing a variational ($\Gamma$-type) limiting behavior of the energy in a limiting regime described by the three model parameters ($Q \to \infty$, $t/dQ^{1/2} \to \infty$, with no assumptions on the ordering of the limits). We identify the energy limit to be a three-dimensional generalization of a functional proposed by Kohn & Müller and investigated by Conti. In the process we need and establish an enhancement of the well-known $\Gamma$-convergence result of Modica & Mortola.

We proceed to use our convergence result to rigorously establish a notion of minimal energy per area (w.r.t. the first two axes) for both the reduced and the full model by considering configurations where this area is finite but tends to infinity. As in the convergence result we do not need to make assumptions about the ordering of the limits. We obtain an asymptotic equality that enhances previous results providing only the scaling behavior up to constants and is novel in the flexibility w.r.t. the parameter limit.

We then turn our attention from the global behavior of the energy to the local energy distribution in a minimizer of the sharp interface model. This provides insight into the domain structure of the minimizer. We show that the energy in a cuboid near the boundary with sufficiently good aspect ratio (between cuboid width and height) scales as if minimizers were self-similar. Indeed, this energy scaling assures that the magnetization $m$ in blow-up sequences converges locally in $L^1$.

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We discuss properties of uniaxially magnetic materials. Our interest is in the energetically optimal state. In this sense we focus on the raw material before it is processed into the ubiquitous modern permanent magnets. We investigate how to connect the well-established micromagnetic Landau-Lifshitz model to experimental observations using rigorous mathematical analysis.

# 1 Introduction

## 1.1 Landau-Lifshitz — a universal model for micromagnetism

A striking feature of the very rich world of micromagnetics is the existence of a widely-accepted universal model encompassing all sorts of effects in a vast range of materials and configurations, the Landau-Lifshitz model.

To start we consider a three-dimensional sample embedded in Euclidean space and denote by $\Omega$ the domain of the sample. We consider a static state. In our semi-classical model the magnetization $m$ is a unit vector field on the sample. For convenience, we extend it by 0 outside $\Omega$ and so consider $m : \mathbb{R}^3 \to \mathbb{R}^3$ with

$$|m|^2 = \begin{cases} 1 & \text{in } \Omega, \\ 0 & \text{elsewhere.} \end{cases}$$

The magnetization induces a stray field $h : \mathbb{R}^3 \to \mathbb{R}^3$. It is determined by the Maxwell equations, greatly simplified by the static nature of our setup to

$$\nabla \cdot (h + m) = 0, \tag{1}$$

$$\nabla \times h = 0. \tag{2}$$

Both $m$ and $h$ are considered to be Lebesgue-measurable vector fields and we identify functions only differing on negligible sets. The differential equations are understood in the sense of distributions on $\mathbb{R}^3$. For notes on the derivation, see e.g. [DKMO05].

At the very heart of the physical model is an energy functional. In absence of an exterior magnetic field, the Lifshitz-Landau energy functional is (after some initial nondimensionalization)

$$E(m) = d^2 \int_\Omega |\nabla m|^2 dx + \int_\Omega \varphi(m) dx + \int_{\mathbb{R}^3} |h|^2 dx.$$

The first term in the sum is called the exchange, the second the anisotropy, and the third the stray field term. Let us give a naïve description of the three terms and introduce $d$ and $\varphi$ on the way.

- The exchange term penalizes spacial variation of $m$. The strength of the term is controlled by the exchange length $d$, a material parameter of typically a few hundred Ångström.

- The anisotropy term enforces a preference for the direction of the magnetization. The function $\varphi : \mathbb{S}^2 \to \mathbb{R}$ is a positive function describing the type and strength of the anisotropy induced by the crystal structure of the material. We shall be interested in $\varphi(m) = Q(m_1^2 + m_2^2)$, making $\pm e_3$ the one *easy axis* and the material *uniaxial*. The dimensionless material parameter $Q$ is called the quality factor.

- Finally, the magnetostatic or stray field energy is the integral of the squared strength of the field induced by the magnetization. Note that the dependence of the field on the magnetization is not a local one. Thus the field extends beyond the sample $\Omega$ and so the domain of integration is $\mathbb{R}^3$ in this term.

With $m$, $h$, $Q$, and $\varphi$ dimensionless and $x$ and $d$ having units of length the units of all three terms of the energy match up as (length)$^3$. The non-locality of the field energy with respect to magnetization changes can be a hassle. We are fortunate enough to be able to trade equation (2) against minimizing $h$ subject to (1). For full detail about the magnetic field we refer the reader to the appendix.

This model when varied with different anisotropy and possibly an additional term representing the influence of an externally applied field explains a vast range of phenomena. In addition many different types of sample geometries can be investigated. Thin films, for example, when $\Omega$ becomes almost two-dimensional have been investigated and simplified two-dimensional models have been established heuristically and through rigorous analysis.

Most appealing to the eye are perhaps the different types of magnetization patterns when made visible by e.g. Kerr microscopy as described in [HS00]. These patterns stem from energy minimization with the nonconvex constraint that the magnetization be of unit length. In the next subsection we specialize to one regime and want to shed some light on the physical effects behind the experimental observations.

Micromagnetics and related phenomena have been extensively studied. On the experimental physics side [HS00] provides an encyclopedic overview of micromagnetic pattern formation phenomena. The survey [DKMO05] compares experimental observations to insights of the mathematical analysis and presents the state of the art (of 2005) in finding mathematically rigorous explanations. The paramount resources for the specific regime under consideration are [CK98] and [CKO99] mentioned above.

## 1.2 Strongly uniaxial ferromagnets

Let us now turn to the specific parameter regime within the Landau-Lifshitz theory that we are interested in, the bulk regime for uniaxial ferromagnets. The push for a good mathematical understanding of this setup and in particular the branching patterns known from physical observation started with [CK98] and [CKO99]. A concise overview of the physical observations, the heuristic explanation for the branching behavior and the energy scaling as well as a short elementary rigorous proof of the lower bound for the scaling can be found in [DKMO05, Chapter 6.8].

As indicated above, we specialize the anisotropy energy contribution to

$$Q \int_\Omega |m'|^2 dx$$

where $m' = \binom{m_1}{m_2}$ is the vector with the first two components of $m$. Our interest lies with *strongly* uniaxial ferromagnets, i.e. we consider very large quality factors

$$Q \gg 1 \qquad \text{(large anisotropy)}.$$

The experimental picture in Figure 1 (more illustrations can be found in [HS00]) shows the domain branching of a neodymium-iron-boron magnet with $Q$ approximately 4. We also consider an idealized sample geometry. To eclipse boundary effects we would like to choose $\Omega = \mathbb{R}^2 \times (-t, t)$. In order to meaningfully talk about energy minimizers in this unbounded sample we introduce some artificial periodicity in the first two coordinates. We thus consider configurations periodic in the first two components $x'$ with fundamental cell $(-l, l)$ for some very large $l$ and then consider the limit $l \uparrow \infty$. Thus we assume that the sample domain is $(-l, l)^2 \times (-t, t)$. The energy then is

$$E_{d,Q,l,t}(m) := d^2 \int_{(-l,l)^2\times(-t,t)} |\nabla m|^2 dx + Q \int_{(-l,l)^2\times(-t,t)} |m'|^2 dx + \int_{(-l,l)\times\mathbb{R}} |h|^2 dx,$$

where $h$ now is a $(-l, l)^2$-periodic solution to the Maxwell equations (1) and (2). Our analysis demonstrates that the energy per cross-section area

$$e(m) := \frac{1}{4l} E(m)$$

converges as $l \to \infty$. This and the fact that the difference between minimal energy with periodic and that with free boundary conditions vanishes in this limit also validate the approach of introducing the artificial periodicity.

It also turns out that the pattern we wish to better understand only forms when $t$ is not too small. We thus consider the regime

$$t \gg dQ^{1/2} \qquad \text{(bulk sample)}.$$

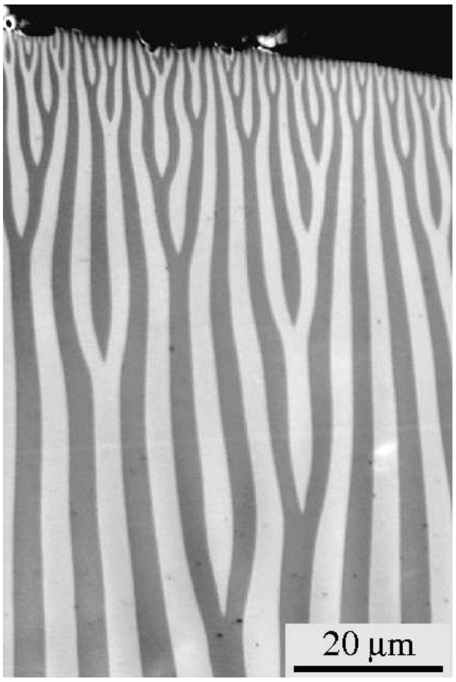


Figure 1: Neodymium-iron-boron magnets with magnetization domains made visible by Kerr microscopy
reproduced from [HS00] with kind permission

The heuristic discussion of Section 2.2 exposes why magnetization patterns with branched domains do indeed achieve low energy and why the cross-over into the bulk regime happens at $t \sim dQ^{1/2}$.

Let us briefly describe the nature of the physical structures: The energy appears to favor the formation of two phases of almost uniform magnetization called *domains*, see Figure 2. These are separated by fairly sharp *walls*, almost lower-dimensional transition regions of a certain *wall width*. In the regime that we are going to study, the domains themselves exhibit a pattern featuring a typical *domain width* when we take a slice parallel to the $x_1x_2$-plane. This domain width is a function of the third coordinate and decreases as the translated planes approach the boundary of the sample and the domains refine by branching. Of particular interest are the limiting domain widths, the *bulk domain width* in the interior and the *surface domain width* at the boundary.

We analyze the magnetic ground state, i.e. the minimizer of $E$ among all $m : \mathbb{R}^3 \to \mathbb{R}^3$ such that $m$ has unit length inside $\Omega$ and vanishes elsewhere.

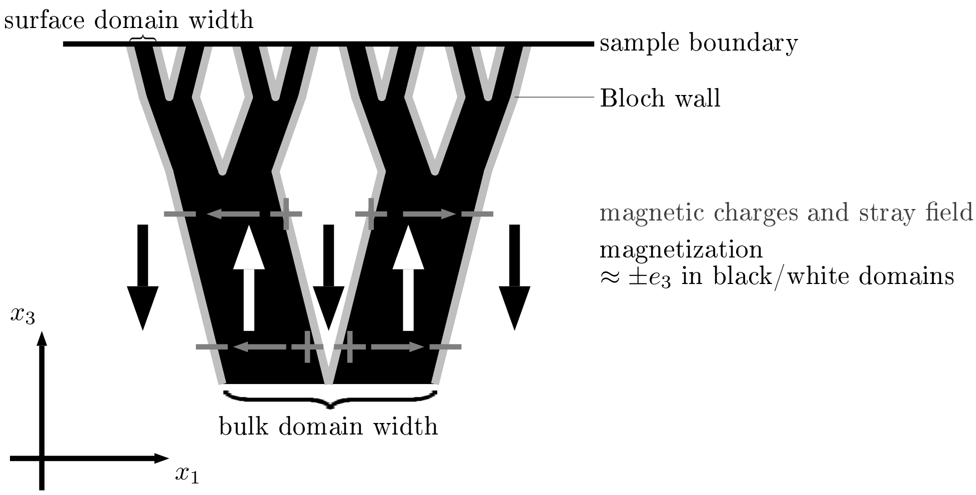

Figure 2: The microstructure in a nutshell

### 1.3 Results

We approach the model with the desire to rigorously establish qualitative properties of ground states such as the formation of domains and the structural refinement of domains. We also want to make quantitative estimates for properties such as the energy, and by proxy of the energy, the domain width in the bulk.

We improve on the upper and lower energy bounds established in [CK98] and [CKO99]. Where the previously known bounds just match in *scaling* in the two non-dimensional parameters, the *quality factor* $Q$ and the quotient of the thickness of the sample by the *exchange length* $t/d$, we show that the appropriately normalized minimal energy per area in $(x_1, x_2)$ converges to a finite universal limit in the parameter regime $t \gg Q^{1/2}d$. This analysis essentially consists of two parts of independent interest.

In the first part we establish a Γ-convergence result on a domain whose lateral size $l$ is large but fixed in terms of the (expected) intrinsic lengthscale of the microstructure, the domain width in the bulk. It is based on an anisotropic rescaling of variables ($x_1$ and $x_2$ are rescaled by the domain width, $x_3$ is rescaled by the thickness $t$).

Interestingly, the Γ-limit turns out to be the 3-d generalization of a functional proposed by Kohn & Müller for twin-branching [KM92, KM94] and investigated by Conti [Con00]. In the rescaled variables, it is given by

$$E_{l,t}(m_3) := 2\int_{[-l,l)^2\times(-t,t)} |\nabla' m_3|\, dx + \int_{(-l,l)^2\times\mathbb{R}} ||\nabla'|^{-1}\partial_3 m_3|^2 dx \tag{3}$$

where again

$$|m_3|^2 = \begin{cases} 1 & \text{for } x_3 \in (-t,t), \\ 0 & \text{otherwise,} \end{cases}$$

and the negative norm of $\partial_3 m_3$ is understood in the sense that

$$\int_{(-l,l)^2\times\mathbb{R}} ||\nabla'|^{-1}\partial_3 m_3|^2 dx := \int_{(-l,l)^2\times\mathbb{R}} |h'|^2 dx$$

where $h' : (-l,l)^2 \times \mathbb{R} \to \mathbb{R}^2$ is the solution to the (reduced) Maxwell equations

$$\begin{aligned} \nabla' \cdot h' + \partial_3 m_3 &= 0, \\ \partial_1 h_2 - \partial_2 h_1 &= 0. \end{aligned}$$

**Theorem 1.** *After a suitable rescaling the reduced energy $E_{l,t}$ is an upper and lower $\Gamma$-type limit of the full energy $E_{d,Q,l,t}$ for fixed rescaled length $l$, $Q \to \infty$, and $(dQ^{1/2}/t)^{1/3} \to 0$.*

We shall make the statement more precise as Theorem 4 before proving it. The subtle part is the construction of a "recovery sequence" in the full parameter regime. It requires a version of the Modica-Mortola construction that is quantitative in the parameter $\frac{\text{wall width}}{\text{domain width}} \ll 1$, since only this quantification allows us to paste this construction into a domain construction which relies on the independently small parameter $\frac{\text{domain width}}{\text{sample thickness}} \ll 1$.

In the second part we show that a notion of *minimal energy per area in the $(x_1, x_2)$-plane* is well-defined in the sense that $l^{-2} \min E$, where the minimum is taken over $m$'s which are $l$-periodic in $(x_1, x_2)$, converges to a finite constant if the artificial "system size" $l$ tends to infinity with respect to the domain width, an intrinsic lengthscale of the microstructure. This means that we establish an extensive behavior reminiscent of the hydrodynamic limits of Ising-type models.

**Theorem 2.** *In the regime of bulk sample and strong anisotropy the minimal energy per surface area*

$$e(Q,d,t,l) = \min\left\{\frac{1}{4l} E_{Q,d,t,l}(m) \,\middle|\, m : \mathbb{R}^3 \to \mathbb{R}^3 \text{ is } (-l,l)^2\text{-periodic in } x', \; |m|^2 = \begin{cases} 1 & \text{for } x_3 \in (-t,t), \\ 0 & \text{otherwise}, \end{cases}\right\}.$$

*is asymptotically proportional to $(dQ^{1/2})^{2/3} t^{1/3}$. More precisely, the limit*

$$\lim_{Q, \frac{t}{dQ^{1/2}}, \frac{l}{(dQ^{1/2})^{1/3} t^{2/3}} \uparrow \infty} \frac{e(Q,d,t,l)}{(dQ^{1/2})^{2/3} t^{1/3}} \in (0,\infty)$$

*exists.*

In other words, the theorem states that there is a universal constant $e^* \in (0,\infty)$ such that for any sequence $\{(d^\nu, Q^\nu, t^\nu)\}_{\nu\in\mathbb{N}} \subset \mathbb{R}^3_+$ satisfying

$$Q^\nu \to \infty, \qquad d^\nu Q^\nu / t^\nu \to 0, \text{ and } \qquad (d^\nu (Q^\nu)^{1/2})^{1/3} (t^\nu)^{2/3} / l \to 0$$

the energy per cross section area behaves as

$$\frac{e(Q^\nu, d^\nu, t^\nu, l)}{(d^\nu (Q^\nu)^{1/2})^{2/3} (t^\nu)^{1/3}} \to e^*.$$

Let us briefly remark that the energy limit when combined with an analogue (which can be shown with the very same proof) of the energy equipartition result of [KM94, Lemma 2.6] to derive an estimate for the total wall energy in the center and thus the typical domain size by considering the area divided by wall length in each slice.

In the third part we analyze the energy distribution in minimizing configurations of the sharp interface model. This is a step beyond the scaling behavior of the energy in the sense that we are actually proving structural properties of a given minimizer. Our method is inspired by work of Conti [Con00] and Alberti, Choksi, and Otto [ACO06] and our result has a form similar to Theorem 2.1 in the former.

**Theorem 3.** *There is a universal constant $C$ such that any $E_{l,1}$-minimizing configuration $m_3$, $h'$ defined on $(-l,l)^2 \times (-1,1)$ and $(-l,l)^2$-periodic in $x'$ has the following property: For any $x'_0 \in (-l,l)^2$, $l_{x_3} \leq 2$, and any $l \geq \lambda \geq l_{x_3}^{2/3} C$*

$$E_{m_3,h'}(x'_0, l_{x_3}, \lambda) := 2 \int_{[-\lambda,\lambda)^2 \times (-1,-1+l_{x_3})} |\nabla' m_3| dx + \int_{(-\lambda,\lambda)^2 \times (-1,-1+l_{x_3})} |h'|^2 dx \sim l_{x_3}^{1/3} \lambda^2.$$

*The constants are universal in the sense that they are independent of $l$, $l_{x_3}$, $x_0$, and $\lambda$.*

We remark that the bound is not expected for small horizontal widths: In constructions and physical observation domain walls have a small angle to the $x_3$-axis in the bulk and the intersection of such a wall with a cylinder would contain interfacial energy of order $\lambda$.

The theorem implies that blowup sequences at sample boundary points have locally convergent ($m_3$ strongly in $L^1$) subsequences.

The application of the theorem with $l_{x_3} = 2$ gives a result that is similar in spirit to Theorem 1.1 of [ACO06], i.e. that on mesoscopic scales the enery is almost uniformly distributed w.r.t. $x'$. The significance of this result becomes apparent when comparing to energy distributions in other models. For minimizers of e.g. the Ginzburg-Landau model for superconductors, almost all energy is localized in small regions of the domain and there are large areas with very little energy.

The key property we use for proving this result is that after localizing the field energy minimizers are locally optimal. This means that energy concentration in a region can only occur when the configuration has high energy at the boundary. We can then “integrate” over these boundaries to get a contradiction with the global energy bounds.

## 1.4 The mathematical vicinity: Patterns and nonconvex variational problems

Microstructures arising in physically motivated energy minimization problems have been prominent in mathematical research for decades. A particularly nice early example from elasticity theory is given by Ball and James in [BJ87] more than twenty years ago. Tellingly, they already have a section entitled *other similar phenomena*, including an account of refinement towards the sample boundary to weakly satisfy boundary conditions that cannot be strongly accomodated. Advancing into the next decade, Müller's lecture notes [Mül99] feature an introduction to the subject of microstructures with theory and examples, including experimental pictures, as well as some notes on the history. The recent introductory lecture [Koh07] given by Kohn at the ICM emphasizes general ideas of pattern formation but also has a taste for micromagnetism. Specializing on magnetics, we have already mentioned DeSimone, Kohn, Müller and Otto's survey [DKMO05]. When Kohn [Koh07] writes “*It should be clear by now that our goal is not to survey the field of energy-driven pattern formation. Such a survey would be extremely difficult, because the subject is vast and ill-defined.*” it seems that similar considerations apply to this thesis introduction. Following Kohn's example we do not attempt to give a complete panorama and instead expose similarities in the analysis in our problem and a physical model of Cahn and Hilliard for phase separation in Section 3. For the broader overview we leave the reader with above starting points for an exploration of the literature. The author learned most of the material of Sections 2 and 3 by lectures of Otto held at the INDAM in Rome and the presentation is heavily based on the lecture notes.

# 2 Heuristics

The goal of this section is to briefly review the heuristic calculations that shed some light on pattern formation, the nature of walls, and estimate a few of the characteristic quantities. These calculations go back to [Hub67], see [HS00, Chapter 3.7.1] for a recent treatment.

We briefly discuss the shape of the domain walls (called Bloch walls), the width of the domain walls, the scaling of domain widths and minimal energy. We heuristically argue that for minimizers

$$\begin{aligned} \text{Bloch wall width} &\sim dQ^{-1/2}, \\ \text{surface domain width} &\sim dQ^{1/2}, \\ \text{bulk domain width} &\sim (dQ^{1/2})^{1/3} t^{2/3}, \text{ and} \\ E(Q,d,t,l) &\sim (dQ^{1/2})^{2/3} t^{1/3} l^2. \end{aligned}$$

Note that the scaling of the domain widths implies (surface domain width) $\ll$ (bulk domain width).

The heuristics for bulk domain width and energy scaling are echoed by the rigorous results of [CKO99].

## 2.1 Bloch walls

Observing the scale separation

$$\text{width of walls} \ll \text{width of domains}$$

we expect domain walls to have a typical profile when cut orthogonally to the direction of the wall. We expect the wall profile to resemble the one-dimensional equilibrium profile.

Let us thus consider magnetizations

$$m = m(x_1) \in \mathbb{S}^2 \text{ such that } \qquad m(\pm\infty) = \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix}.$$

We estimate (a bit on the formal side) the sum of exchange and anisotropy energy as

$$\begin{aligned} \int_{-\infty}^{\infty} d^2 \left|\frac{dm}{dx'}\right|^2 + Q|m'|^2 dx_1 &\geq \int_{-\infty}^{\infty} d^2 \left(\frac{d|m'|}{dx_1}\right)^2 + d^2 \left(\frac{dm_3}{dx_1}\right)^2 + Q|m'|^2 dx_1 \\ &= \int_{-\infty}^{\infty} d^2 \left(\frac{d|m'|}{dx_1}\right)^2 + d^2 \left(\frac{dm_3}{dx_1}\right)^2 + Q(1-m_3^2) dx_1 \\ &= \int_{-\infty}^{\infty} d^2 \frac{m_3^2}{1-m_3^2} \left(\frac{dm_3}{dx_1}\right)^2 + d^2 \left(\frac{dm_3}{dx_1}\right)^2 + Q(1-m_3^2) dx_1 \\ &= \int_{-\infty}^{\infty} d^2 \frac{1}{1-m_3^2} \left(\frac{dm_3}{dx_1}\right)^2 + Q(1-m_3^2) dx_1 \\ &\geq 2dQ^{1/2} \int_{-\infty}^{\infty} \frac{dm_3}{dx_1} dx_1 = 2dQ^{1/2}. \end{aligned}$$

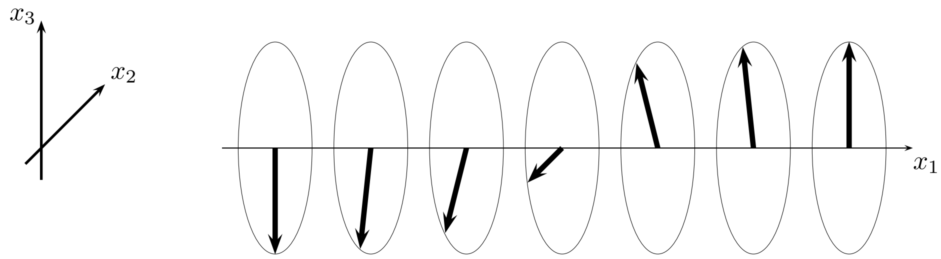


Figure 3: 1-dimensional Bloch wall profile, the unit vector $m(x_1)$ rotates in the $x_2x_3$-plane

The first inequality turns into an equation if and only if $\frac{dm'}{dx_1}$ is parallel to $m'$. But then, the magnetization (locally) assumes values only in one axis of the $m'$-plane and we can avoid magnetic charges (and thus field energy) entirely by setting $m_1 \equiv 0$. Equality in the Cauchy-Schwarz-inequality of the last line is achieved if and only if the two summands are equal, i.e.

$$\frac{dm_3}{dx_1} = \frac{Q^{1/2}}{d}(1 - m_3^2).$$

It is no secret that the (unique up to translation in $x_1$) solution to this ODE with above boundary conditions is

$$m_3(x_1) = \tanh\Big(\frac{Q^{1/2}}{d}x_1\Big).$$

We complement this $m_3$ with $m'$ pointing in, say, the $x_2$ direction and length such that $|m|^2 = 1$, see Figure 3. This type of domain wall is called a *Bloch wall.* Note that this function has the fairly steep slope $d^{-1}Q^{1/2}$ at the origin and then approaches $\pm 1$ quickly with the distance to the origin measured in units of $dQ^{-1/2}$. While it is of limited use to speak about an exact wall width this behavior certainly justifies the scaling relation

$$\text{Bloch wall width} \sim dQ^{-1/2}.$$

In terms of energy we expect

$$\frac{\text{Bloch wall energy}}{\text{wall area}} \approx 2dQ^{1/2}.$$

## 2.2 The energetic advantage of domain branching

To get some taste for why domains form branched patterns we investigate three types of possible magnetization patterns, uniform magnetization, striped, and branched domains (also see Figure 2). All three magnetization patterns are constant in one horizontal direction, it later turns out that this is sufficient to achieve the optimal energy scaling. More information can be found in [Hub67] and [CK98].

Recall that

$$E(m) = d^2 \int_{(-l,l)^2\times(-t,t)} |\nabla m|^2 dx + Q \int_{(-l,l)^2\times(-t,t)} |m'|^2 dx + \int_{(-l,l)^2\times\mathbb{R}} |h|^2 dx,$$

where $\nabla\cdot(h+m) = 0$ and $\nabla\times h = 0$.

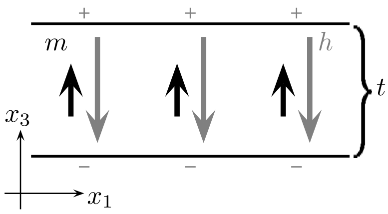

Figure 4: Uniform magnetization

The first, simplest ansatz is to use constant magnetization in vertical orientation $m \equiv e_3$. This has to be compensated by a field $h = -\chi_{(-l,l)^2\times(-t,t)} e_3$ and so

$$d^2 \int |\nabla m|^2 dx = 0, \quad Q \int |m'|^2 dx = 0, \quad \int |h|^2 dx = 8l^2 t$$

so that $E = 8l^2 t$, i.e. the stray field energy is rather large. We can now try to reduce the stray field energy at the expense of introducing walls.

For the second ansatz, depicted in Figure 5, we thus use a vertical magnetization that is constant in strips of width $w \ll t$ with alternating orientation between neighboring strips. In this scenario we get

$$d^2 \int |\nabla m|^2 dx + Q \int |m'|^2 dx \approx 2Q^{1/2} d\,(\text{wall area}) \sim Q^{1/2} d \frac{l^2 t}{w}.$$

The field energy necessarily scales like $l^2 w$ by variable transformation, but the corresponding magnetic field can also be easily constructed as the gradient of a potential pieced together from summands of the form

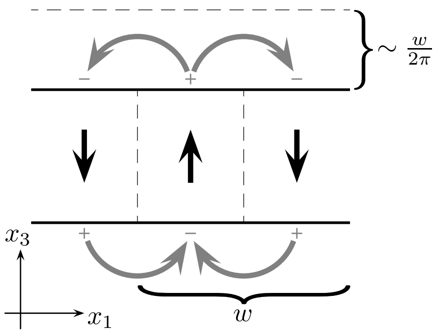

Figure 5: Alternating magnetization

$$u^{(k)} = \frac{4}{\pi(2k+1)} \sin\left((2k+1)\frac{2\pi}{w} x_1\right) \cdot \frac{w}{2\pi(2k+1)} e^{-(2k+1)\frac{2\pi}{w}(x_3 - t)}$$

on the upper half-space outside the sample and similarly below the sample. The field energy in each strip on a $x_2$-slice can then be computed as

$$\int_0^w \int_t^\infty |\nabla(\sum_{k\geq 0} u^{(k)})|^2 dx_3 dx_1 \sim w^2$$

(note that $\nabla u^{(k)}$ are $L^2$-orthogonal because their $x_1$-dependent parts are). Note that the decay of the field is exponentially fast away from the sample boundary with distance measured in units of $\frac{w}{2\pi}$. One way or the other

$$\int |h|^2 dx \sim l^2 w$$

so that

$$E \sim Q^{1/2} d \frac{l^2 t}{w} + l^2 w$$

and with the optimal segment width $w = d^{1/2} Q^{1/4} t^{1/2}$ the energy is

$$E \sim l^2 d^{1/2} Q^{1/4} t^{1/2}.$$

Thus we see that this configuration is energetically better than uniform magnetization if $t \gg dQ^{1/2}$.

We observe that in the striped pattern magnetic charges inducing the stray field only occur at the sample boundaries. If we can have a smaller width $w$ there without having to add as many walls cutting all the way through the sample, we could do even better. For our third ansatz we thus consider branched domains to reduce the stray even more. The philosophy is to let the characteristic width of the domain $w$ vary with $x_3$, so that close to the edge, $w$ can be small to minimize the magnetostatic energy, and in the bulk, $w$ can be large to minimize wall energy.

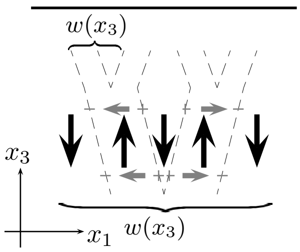

Figure 6: Branched magnetization

Schematically, the branching has the form depicted in Figure 6. If we neglect the area increase of the domain walls caused by the tilt we can estimate

$$\int_0^{\Delta t} \int d^2|\nabla m|^2 + Q|m'|^2 dx' dx_3 \approx 2dQ^{1/2}\text{wall area} \sim dQ^{1/2}\Delta t \frac{l^2}{w}.$$

In fact, in the reduced model that we derive for large $Q$ the interfacial energy is proportional to the $x_3$-slicewise wall area and in particular the tilt does not play a role. The tilting of the interfaces causes the magnetization to change in the $x_3$-direction. This induces magnetic charges and a field to accomodate them. The field strength is proportional to the tilt $w/\Delta t$ and its support has area $\sim \Delta t l^2$. Thus the field energy scales like

$$\int_0^{\Delta t} \int |h|^2 dx' dx_3 \sim \Delta t l^2 \left(\frac{w}{\Delta t}\right)^2.$$

To compute the total energy, we need to compose multiple layers. Expecting again the field and interface energy to be balanced for low-energy configurations (also see e.g. [CKO99, Proposition 4.1] for a rigorous result in this direction), the natural scaling is the one that keeps this balance, i.e.

$$w \sim \left(dQ^{1/2}\right)^{1/3} \Delta t^{2/3}. \tag{4}$$

In order to obtain a dyadic refinement of $w$ we choose a sequence of $\Delta t \sim 2^{-3k/2}$ with the constant chosen so that one set of refined layers covers $(0,t)$ and another $(-t,0)$. With the above calculation of the energy for one iteration of the branching we then see that the total energy scales according to

$$\begin{aligned} E &\sim \sum_{\Delta t} \left( dQ^{1/2}\Delta t \frac{l^2}{w} + \Delta t l^2 \left(\frac{w}{\Delta t}\right)^2 \right) \\ &\overset{(4)}{\sim} \sum_{\Delta t} l^2 (\Delta t)^{1/3} (dQ^{1/2})^{2/3} \\ &\sim t^{1/3} l^2 (dQ^{1/2}), \end{aligned}$$

much better (in the regime $t \gg dQ^{1/2}$) than the simple striped pattern. Note that in the center (and more generally away from the boundary), the typical domain width is $w \sim (dQ^{1/2})^{1/3} t^{2/3}$.

To sum up, we see a crossover at $t \sim dQ^{1/2}$ in the sense that of the three configuration investigated here, the branched is best for $t \gg dQ^{1/2}$ while the uniform magnetization is best for $t \ll dQ^{1/2}$. The first regime is precisely what we call the *bulk regime*. It can be shown that these are the optimal scalings. For the bulk regime this is done in Lemma 5.

## 2.3 Domain width at the surface

Let us briefly take a look – on the level of heuristical calculation – at the surface domain width. Starting from the energy

$$E(m) = d^2 \int_{(-l,l)^2\times(-t,t)} |\nabla m|^2 dx + Q \int_{(-l,l)^2\times(-t,t)} |m'|^2 dx + \int_{(-l,l)^2\times\mathbb{R}} |h|^2 dx$$

we can heuristically express it in terms of the domain width $w(x_3)$ by splitting the stray field into bulk and surface effects and doing Modica-Mortola-style contraction of the first two terms

$$\begin{aligned} E &\sim dQ^{1/2} \int_{[-l,l)^2\times(-t,t)} |\nabla' m_3| dx \\ &\quad + \int_{(-l,l)\times(-t,t)} ||\nabla'|^{-1}\partial_3 m_3|^2 dx + \sum_{x_3=\pm t} \int_{(-l,l)^2} ||\nabla'|^{-1/2} m_3|^2 dx' \\ &\sim dQ^{1/2} \int_{-t}^{t} \frac{1}{w} dx_3 + \int_{-t}^{t} (\partial_3 w)^2 dx_3 + w(t) + w(-t). \end{aligned}$$

Computing the first variation we find the Euler-Lagrange-Equation

$$-\frac{dQ^{1/2}}{w^2} - 2\partial_3^2 w = 0 \tag{5}$$

with boundary conditioons

$$2\partial_3 w(\pm t) \pm 1 = 0.$$

The first integral of (5) is found by multiplying with $\partial_3 w_3$ and rewriting as

$$\partial_3\big(\frac{dQ^{1/2}}{w} - (\partial_3 w)^2\big) = 0$$

whence

$$(\partial_3 w)^2 = \frac{dQ^{1/2}}{w} - \frac{dQ^{1/2}}{w(0)}.$$

The integration constant has been determined using $\partial_3 w = 0$ grace à the symmetry of $w$. Plugging in the boundary condition and our expectation that $w(0)$ is larger than $w(t)$ we see that

$$\frac{1}{4} = (\partial_3 w(t))^2 = \frac{dQ^{1/2}}{w(t)} - \frac{dQ^{1/2}}{w(0)} \approx \frac{dQ^{1/2}}{w(t)},$$

in other words

$$w(t) \sim dQ^{1/2},$$

is the surface domain width.

We also see that if we are in the regime $dQ^{1/2} \ll t$ and rescale $t$ to 1, the surface domain width becomes very small. In the limiting model the domains completely refine towards the boundary. For large but finite ratios, the refinement stops at very small distances from the boundary. This is detailed in [HS00, Ch. 3.7.5].

# 3 Different physics – similar mathematics: Coarsening in the Cahn-Hilliard model

We briefly introduce the Cahn-Hilliard model for phase transitions in order to expose the parallels to our model with respect to lower energy bounds. This model, also known as *Model B* in the physics literature, is used to describe the evolution of microstructures in mixtures, e.g. during the cooling of alloys.

The crucial quantity in this model is a scalar order parameter $u : (0,\Lambda)^d \to \mathbb{R}$. We assume that $u$ is periodic with fundamental cell $(0,\Lambda)^2$. While the range of solutions $u$ is not necessarily bounded the physical interpretation is that $u$ typically takes values in $[0,1]$ and is the density of one component or phase in a two-component alloy. Throughout this section we also use the corresponding sharp interface version, the Mullins-Sekerka model. The simplified, purely interfacial energy is motivated in Section 3.1. In this model $u$ is mandated to take values in $\{0,1\}$.

But first let us very briefly motivate the derivation of the Cahn-Hilliard equation. We introduce the Ginzburg-Landau energy

$$E = \fint e dx, \qquad e = \frac{1}{2}|\nabla u|^2 + \frac{1}{2}(u(1-u))^2.$$

Here and in the following we use the notation $\fint \cdot dx = \frac{1}{\Lambda^d}\int_{(0,\Lambda)^d} \cdot dx$.

The relaxation of the energy with conservation of the average $\Phi := \fint u dx$ of the order parameter is realized by

$$\dot{u} - \Delta\frac{\partial e}{\partial u} = 0, \qquad \frac{\partial e}{\partial u} = -\Delta u + u(1-u)(1-2u). \tag{6}$$

Indeed,

$$\begin{aligned} \dot{\Phi} &= \fint \dot{u} dx = \fint \Delta\frac{\partial e}{\partial u} dx = 0, \\ \dot{E} &= \fint \frac{\partial e}{\partial u}\dot{u} dx = \fint \frac{\partial e}{\partial u}\Delta\frac{\partial e}{\partial u} dx = -\fint |\nabla\frac{\partial e}{\partial u}|^2 dx = -\fint ||\nabla|^{-1}\dot{u}|^2 dx, \end{aligned}$$

since $\Delta\frac{\partial e}{\partial u} = \dot{u}$.

The Cahn-Hilliard evolution as defined by (6) is the gradient flow of $E$ with respect to the Euclidean structure given by $\||\nabla|^{-1} \cdot \|_{L^2}$.

Mathematically the order parameter $u$ plays a rôle similar to that of the third component $m_3$ of the magnetization. Both the magnetic and the Cahn-Hilliard model have an interpolation inequality at the core of the argument for an energy bound: For the magnetic problem in Lemma 4 and for Mullins-Sekerka in Lemma 1 and Lemma 2 for the sharp-interface. For the original Cahn-Hilliard model Propositions 2 and 3 are of the same nature even though they do not take the form of a classical interpolation inequality. These inequalities are given a physical interpretation when we bound the problem-specific parameter with nonconvex constraint (e.g. $m_3 \in \{-1,+1\}$ in magnetics, $u \in \{0,1\}$ in Mullins-Sekerka) by the norm of $u$ on the left hand side and then interpret the two terms on the right hand side as interfacial and (e.g. magnetic or diffusion) field energy, respectively.

These interpolation inequalities are then used to derive bounds on the energy. For magnetics this is done in Lemma 5. For Cahn-Hilliard and Mullins-Sekerka we use Proposition 1 from [ORS06] which provides a generic framework for gradient flows using Otto's formal Riemannian calculus. In the micromagnetic case the interpolation inequality we need is two-dimensional (in $x'$) and then integrated over the third component. Similarly, interpolation inequalities in space lead to a lower bound for the time-integral of the energy for the gradient flow.

A lot of pioneering work in the exploration of this connection of physical phenomena to inequalities in mathematical analysis has been done by Kohn and Otto, a particular starting point is [KO02].

## 3.1 Interfacial regime, heuristics

We wish to heuristically calculate the energetic behavior of $u$ in the vicinity of an interface. Guided by physical observations we postulate a separation of scales, namely

thickness of interface layer $\ll$ radius of curvature of interface layer.

Under this assumption, we expect the transition layer to take the shape of a one-dimensional equilibrium profile. This can be computed by considering the energy minimization amongst all $u$ such that $u(-\infty) = 0$ and $u(+\infty) = 1$ of

$$E(u) = \int_{-\infty}^{\infty} \frac{1}{2}\left(\frac{du}{dx}\right)^2 + \frac{1}{2}(u(1-u))^2 dx \geq \int_{-\infty}^{\infty} \left(\frac{du}{dx}\right)(u(1-u))dx = \int_0^1 u(1-u)du = \frac{1}{6}.$$

Note that equality is attained if and only if $\frac{du}{dx} = u(1-u)$.

As a consequence we expect the interfacial energy to be roughly

$$E \approx \frac{1}{6}\frac{\text{area of interface layer}}{\text{volume of system}}. \tag{7}$$

We also see that the one-dimensional case is special in that the energy is approximately

$$E \approx \frac{1}{6} \text{ number density of kinks.}$$

Thus the system is coarsening very slowly because the interfacial energy is only reduced when the number of kinks decreases. The exact behavior has been investigated by Carr and Pego [CP89].

## 3.2 Coarsening for Cahn-Hilliard, $\Phi \ll 1$, heuristics

Our strategy is to exploit that the decay rate of friction is the energy dissipation by evaporation and recondensation, i.e.

$$\dot{E} = -⨍ \left||\nabla|^{-1}\dot{u}\right|^2 dx.$$

To obtain a heuristic ansatz, we observe the following.

- In the low volume fraction regime particles are almost spherical. Thus, their form is essentially controlled by the average radius $R(t)$.
- The particles do not move (but can vanish), thus the average distance $L$ between particles is a typical quantity.

Our goal is now to express $\Phi$, $E$, and $⨍ \left||\nabla|^{-1}\dot{u}\right|^2 dx$ in terms of $L$, $R$, and $\dot{R}$ to derive an evolution for those. To this end we notice that

$$\Phi = ⨍ u dx = \text{ number density } \times \text{ volume of individual particle} \sim L^{-d}R^d.$$

For the energy we see with our previous consideration (7) that

$$E \approx \frac{1}{6}\frac{\text{area of interface layer}}{\text{volume of system}} \approx \frac{1}{6} \text{ number density of particles } \times \text{ average surface of particle}$$
$$\sim L^{-d}R^{d-1} \sim \Phi R^{-1}.$$

Finally, we calculate that

$$\fint \left||\nabla|^{-1}\dot{u}\right|^2 dx = \fint |\nabla v|^2 dx = \fint |J|^2 dx,$$

where $v$ is the potential of $\dot{u}$, i.e. $-\Delta v = \dot{u}$, and $J$ is the diffusion flux given by $\nabla \cdot J = \dot{u}$.

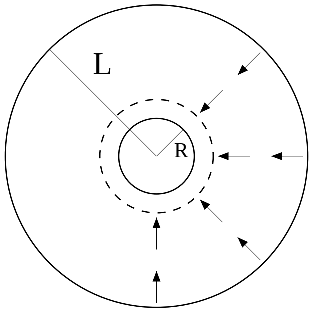


Figure 7: Neighborhood of a typical particle

Assuming $\dot{R} > 0$, we see that

$$J \approx \frac{\vec{r}}{r^d}\frac{\text{change of volume of particle}}{\text{area of } \mathbb{S}^{d-1}}$$
$$\sim \frac{\vec{r}}{r^d}(R^d)^{\cdot},$$

thus

$$\fint |J|^2 dx \approx \text{ number density } \times \int_R^L \left|\frac{\vec{r}}{r^d}(R^d)^{\cdot}\right|^2 r^{d-1} dr$$
$$\sim L^{-d}\left((R^d)^{\cdot}\right)^2 \int_R^L r^{-d+1} dr$$
$$\sim L^{-d}R^{2d-2}(\dot{R})^2 \begin{cases} R^{-d+2} & \text{if } d \geq 3, \\ \ln \frac{L}{R} & \text{if } d = 2, \end{cases}$$
$$\sim \Phi(\dot{R})^2 \begin{cases} 1 & \text{if } d \geq 3, \\ \ln \frac{1}{\Phi} & \text{if } d = 2. \end{cases}$$

Hence $E \sim \Phi R^{-1}$ and $\dot{E} = -\fint \left||\nabla|^{-1}\dot{u}\right|^2 dx$ can be combined to yield

$$-\Phi R^{-2}\dot{R} = (\Phi R^{-1})^{\cdot} \sim -\Phi(\dot{R})^2 \begin{cases} 1 & \text{if } d \geq 3, \\ \ln \frac{1}{\Phi} & \text{if } d = 2. \end{cases}$$

Solving for $R$ we find

$$\left(R^3\right)^{\cdot} \sim R^2\dot{R} \sim \begin{cases} 1 & \text{if } d \geq 3, \\ \ln^{-1} \frac{1}{\Phi} & \text{if } d = 2, \end{cases}$$

and thus

$$R \sim t^{1/3} \begin{cases} 1 & \text{if } d \geq 3, \\ \ln^{-1/3} \frac{1}{\Phi} & \text{if } d = 2 \end{cases}$$

and

$$E \sim \Phi R^{-1} \sim t^{-1/3} \begin{cases} \Phi & \text{if } d \geq 3, \\ \Phi \ln^{1/3} \frac{1}{\Phi} & \text{if } d = 2. \end{cases}$$

The question arises if and how the heuristically derived behavior can be proven rigorously.

- The answer is *no* for the upper bound on the energy

$$E \lesssim t^{-1/3} \begin{cases} \Phi & \text{if } d \geq 3, \\ \Phi \ln^{1/3} \frac{1}{\Phi} & \text{if } d = 2. \end{cases}$$

  Such a bound cannot be obtained unconditionally because there exist ungeneric solutions that do not coarsen at all or only very slowly.

- The answer is *yes* for a lower energy bound, at least in a time-averaged sense, as we see in the remainder of this section.

## 3.3 Abstract framework for lower bounds

The abstract framework proposes a philosophy for the gradient flow where the energy landscape determines the dynamics. The following proposition links the exponent in an estimate describing the geometry of the energy landscape to the dynamic exponent in the lower bound of the energy. The ideas presented here are properly developed and discussed in greater depth in the work of Kohn & Otto [KO02] and Otto, Rump & Slepčev [ORS06].

**Proposition 1** ([ORS06]). *Let $X$ be an affine space, $u_0 \in X$, $E : X \to [0, \infty]$, $E_0 \geq 0$, $\alpha \geq 0$, and*

$$E(u) \geq E_0 |u - u_0|^{-\alpha} \text{ for all } u \in X \text{ with } E(u) \leq E_1.$$

*Then for any $\sigma \in (1, 1 + \frac{2}{\alpha})$ and any solution of*

$$\dot{u} = -gradE(u)$$

*we have*

$$\int_0^T E(u(t))^\sigma dt \geq C(0) \int_0^T (E^{\frac{2}{2+\alpha}} t^{-\frac{\alpha}{\alpha+2}})^\sigma dt \tag{8}$$

*provided $T \geq E_0^{-1} |u(+\infty) - u_0|^{\alpha+2}$ and $E(u(+\infty)) \leq E_1$.*

*Remark* 1. Estimate (8) is a time-averaged version of

$$E(u(t)) \gtrsim E_0^{\frac{2}{2+\alpha}} t^{-\frac{\alpha}{\alpha+2}},$$

and $\frac{\alpha}{\alpha+2}$ is the geometric exponent alluded to above.

The pointwise (in time) estimate does not follow from the assumptions of the proposition, this is discussed in [ORS06, Remark 2].

*Remark* 2. To motivate the connection between the geometric exponent $\alpha$ and the dynamic exponent $\frac{\alpha}{\alpha+2}$, let us consider the simple example of $X = \mathbb{R}$, $u_0 = 0$, and $E(u) = E_0 |u|^{-\alpha}$. To avoid tracking the sign we consider a point where $u > 0$. Then $\operatorname{grad} E(u) = -\alpha E_0 u^{-\alpha-1}$, so that $\dot{u} = -\operatorname{grad} E(u)$ turns into $\dot{u} = \alpha E_0 u^{-\alpha-1}$ or equivalently $(u^{\alpha+2})^{\cdot} = \alpha(\alpha+2)E_0$. Thus

$$u^{\alpha+2} = \alpha(\alpha+2)E_0 t + u(t=0)^{\alpha+2}$$

and for $t \gg E_0^{-1} u(t=0)^{\alpha+2}$

$$E(u) = E_0(\alpha(\alpha+2)E_0 t + u(t=0)^{\alpha+2})^{-\frac{\alpha}{\alpha+2}} \sim E_0^{\frac{2}{\alpha+2}} t^{-\frac{\alpha}{\alpha+2}}.$$

### 3.4 Interpolation inequalities and lower energy bounds ($d \geq 3$)

In this section we proceed to obtain the first set of energy bounds announced at the beginning of Section 3. To this end we want to apply Proposition 1 to

- The space $X = \{u : (0,\Lambda)^d \to \mathbb{R} \text{ periodic}, \fint u\,dx = \Phi\}$ equipped with the inner product from $\|\cdot\| = \left(\fint \left||\nabla|^{-1}(u-\Phi)\right|^2 dx\right)^{1/2}$,
- $u_0 \equiv \Phi$, so that $|u - u_0| \leq \|u\|$, and
- $E(u) = \fint \frac{1}{2}|\nabla u|^2 + \frac{1}{2}(u(1-u))^2 dx$.

We thus need to find an optimal $\alpha$ and $E_0$ with optimal scaling in $\Phi$ such that

$$\fint \frac{1}{2}|\nabla u|^2 + \frac{1}{2}(u(1-u))^2 dx \geq E_0 \left(\left(\fint \left||\nabla|^{-1}(u-\Phi)\right|^2 dx\right)^{1/2}\right)^{-\alpha}.$$

For simplicity we resort to the interfacial regime approximation introduced in Section 3.1 and so replace

$$E = \fint \frac{1}{2}|\nabla u|^2 + \frac{1}{2}(u(1-u))^2 dx, \qquad u \in \mathbb{R}$$

with

$$E = \fint \frac{1}{6}|\nabla u| dx = \frac{1}{6}\frac{\text{area of } \partial\{u=1\}}{\text{volume of system}}, \qquad u \in \{0,1\},$$

i.e. the energy is the perimeter of the set where $u = 1$, and our task becomes to find $\alpha$ and $E_0$ with optimal scaling in $\Phi$ such that

$$\fint |\nabla u| dx \geq E_0 \left(\left(\fint \left||\nabla|^{-1}(u-\Phi)\right|^2 dx\right)^{1/2}\right)^{-\alpha}$$

for $u : (0,\Lambda)^d \to \mathbb{R}$ periodic with $\fint u dx = \Phi$. The bound presented in this section exhibits optimality for $d \geq 3$ only. The improvement necessary for optimal scaling when $d = 2$ is accomplished in the next section.

**Lemma 1.** *For any $d \in \mathbb{N}$ there is a $C \leq \infty$ such that for all $w : (0,\Lambda)^d \to \mathbb{R}$ with $\fint w dx = 0$ we have*

$$\|w\|_{wL^{4/3}} := \sup_{s>0} s|\{|w| \geq s\}|^{3/4} \leq C\|\nabla w\|_{L^1}^{1/2} \||\nabla|^{-1} w\|_{L^2}^{1/2}.$$

Before proceeding to the proof, we want to illustrate how to apply Lemma 1 to obtain the geometric control required in Proposition 1. For this, we plug in $w = u - \Phi$ and $s = \frac{1}{2}$ to see that

$$|\{|u-\Phi| \geq \frac{1}{2}\}|^{3/4} \lesssim \|\nabla u\|_{L^1}^{1/2} \||\nabla|^{-1}(u-\Phi)\|_{L^2}^{1/2}$$

and thus

$$\begin{aligned}\left(\fint |\nabla u| dx\right)^{1/2} \left(\fint ||\nabla|^{-1}(u-\Phi)|^2 dx\right)^{1/4} &\gtrsim \fint\{|u-\Phi| \geq \frac{1}{2}\}|^{3/4} \geq \fint\{u \geq \frac{1}{2} + \Phi\}|^{3/4} \\ &\geq \fint\{u = 1\}|^{3/4} = \Phi^{3/4},\end{aligned}$$

where we denote the volume fraction of a set by $⨍ . |$ and use that the volume fraction of $\{u=1\}$ is $\Phi \ll 1$. We thus obtain

$$⨍ |\nabla u| dx \geq \frac{1}{C}\Phi^{3/2}\left(\left(⨍ ||\nabla|^{-1}(u-\Phi)|^2 dx\right)^{1/2}\right)^{-1}.$$

For comparison with the heuristics above, we use this estimate in Proposition 1 with $\alpha = 1$ and $E_0 = \Phi^{3/2}$. We obtain in a time-averaged sense

$$E(u(t)) \gtrsim (\Phi^{3/2})^{\frac{2}{2+1}} t^{-\frac{1}{1+2}} = \Phi t^{-\frac{1}{3}},$$

which is the result predicted heuristically.

*Proof of Lemma 1.* By scaling, we may assume $s = 1$. Consider the function

$$\chi = \begin{cases} 1 & \text{where} \quad 1 \leq w, \\ 0 & \text{where} \ -1 < w < 1, \text{ and} \\ -1 & \text{where} \qquad w \leq -1, \end{cases}$$

select a smooth, compactly supported, symmetric, nonnegative convolution kernel $K_1$ with $\int K_1 dz = 1$ and define $K_R(z) = \frac{1}{R^d} K_1\left(\frac{z}{r}\right)$.

Now

$$\begin{aligned} \int |\chi| dx \leq \int \chi w dx &= \int (w - K_R * w)\chi dx + \int (K_R * w)\chi dx \\ &= \int (w - K_R * w)\chi dx + \int w (K_R * \chi) dx \\ &\leq \int |w - K_r * w| dx \sup|\chi| + \left(\int \left||\nabla|^{-1} w\right|^2 dx\right)^{1/2} \left(\int |\nabla (K_R * \chi)|^2 dx\right)^{1/2}. \end{aligned}$$

As the terms are bounded by $\int |w - K_r * w| dx \lesssim R \int |\nabla w| dx$, $\sup |\chi| \leq 1$, and

$$\int |\nabla (K_R * \chi)|^2 dx \leq \left(\int |\nabla K_R| dx\right)^2 \int \chi^2 dx \lesssim R^{-2} \int |\chi| dx,$$

we arrive at

$$\int |\chi| dx \lesssim R \int |\nabla w| dx + \left(\frac{1}{R^2}\int \left||\nabla|^{-1} w\right|^2 dx \int \chi dx\right)^{1/2}.$$

Using Young's inequality, we can absorb the (finite) rightmost integral and get

$$\int |\chi| dx \lesssim R \int |\nabla w| dx + \frac{1}{R^2}\int \left||\nabla|^{-1} w\right|^2 dx,$$

so that with the optimal choice of $R = \frac{(\int ||\nabla|^{-1} w|^2 dx)^{1/3}}{(\int |\nabla w| dx)^{1/3}}$,

$$|\{|w| \geq 1\}| = \int |\chi| dx \lesssim \left(\int |\nabla w| dx\right)^{2/3} \left(\int \left||\nabla|^{-1} w\right|^2 dx\right)^{1/3}.$$

Hence

$$|\{|w| \geq 1\}|^{3/4} = \int |\chi| dx \lesssim \left(\int |\nabla w| dx\right)^{1/2} \left(\int \left||\nabla|^{-1} w\right|^2 dx\right)^{1/4},$$

as claimed. □

With essentially the same techniques as above we can obtain the analogous result for the full Cahn-Hilliard energy.

**Proposition 2.** *For any $d \in \mathbb{N}$ there exists a constant $C \leq \infty$ such that for all $\Lambda > 0$ and $u : (0,\Lambda)^d \to \mathbb{R}$ with $\Phi = ⨍ u dx \ll 1$ satisfying*

$$E := ⨍ \frac{1}{2}|\nabla u|^2 + \frac{1}{2}(u(1-u))^2 dx \leq \frac{1}{C}\Phi^2$$

*we have*

$$E \geq \frac{1}{C}\Phi^{3/2}\left(⨍ \left||\nabla|^{-1}(u-\Phi)\right|^2\right)^{-1/2}.$$

The assumption $E \ll \Phi^2$ ensures that $u$ has evolved well into the interfacial regime (compare with assumption $E \leq E_1$ in the abstract framework described by Proposition 1).

*Proof.* Compared to Lemma 1, we need the new estimates

$$\begin{aligned}
⨍_{\{\frac{1}{3}\leq u\leq\frac{2}{3}\}} |\nabla u| dx &\lesssim ⨍ |u(1-u)|\,|\nabla u| dx \\
&\leq ⨍ \frac{1}{2}|\nabla u|^2 + \frac{1}{2}(u(1-u))^2 dx = E, \\
⨍_{\{u\leq -1\}\cup\{u\geq 2\}} |u| &\lesssim ⨍ \frac{1}{2}(u(1-u))^2 dx \leq E, \text{ and} \\
\Phi = ⨍ u dx &\lesssim ⨍ |u(1-u)| dx + ⨍\{u \geq \frac{2}{3}\}| \\
&\lesssim \left(⨍ \frac{1}{2}(u(1-u))^2 dx\right)^{1/2} + ⨍\{u \geq \frac{2}{3}\}| \\
&\leq E^{1/2} + ⨍\{u \geq \frac{2}{3}\}|.
\end{aligned} \tag{9}$$

Here and in the following, we write $⨍ := \Lambda^{-d}\int$, using "volume" $\Lambda^d$ in the denominator regardless of the integration domain.

We now introduce the cut-off function

$$\chi = \begin{cases} 1 & \frac{2}{3} \leq u, \\ 3u-1 & \frac{1}{3} \leq u \leq \frac{2}{3}, \\ 0 & u \leq \frac{1}{3}. \end{cases}$$

Similar to the calculation in Lemma 1 we have

$$\begin{aligned}
\fint \chi\,dx &\lesssim \fint u\chi\,dx \\
&= \fint uK_R * \chi dx + \fint u(\chi - K_R * \chi)dx \\
&= \fint (u-\Phi)K_R * \chi dx + \Phi \fint K_R * \chi dx \\
&\quad + \fint_{\{-1\le u\le 2\}} u(\chi - K_R * \chi)dx + \fint_{\{u<-1\}\cup\{u>2\}} u(\chi - K_R * \chi)dx \\
&\le \left(\fint ||\nabla|^{-1}(u-\Phi)|^2 dx \fint |\nabla(K_R * \chi)|^2 dx\right)^{1/2} + \Phi \int K_R dx \fint \chi\,dx \\
&\quad + 2\fint |\chi - K_R * \chi| dx + \fint_{\{u<-1\}\cup\{u>2\}} |u| dx \sup_x |\chi - K_R * \chi|.
\end{aligned}$$

Using

$$\begin{aligned}
\fint |\nabla(K_R * \chi)|^2 dx \quad &\lesssim \quad \frac{1}{R^2}\fint \chi dx, \\
\int K_R dx \quad &= \quad 1, \\
\Phi \quad &\ll \quad 1, \\
\fint |\chi - K_R * \chi| dx \quad \lesssim \quad R\fint |\nabla\chi| dx \quad &= \quad 3R\int_{\{\frac{1}{3}\le u\le\frac{2}{3}\}} |\nabla u| dx \quad \lesssim \quad RE, \\
\fint_{\{u<-1\}\cup\{u>2\}} |u| dx \quad &\lesssim \quad E, \text{ and} \\
\sup|\chi - K_R * \chi| \quad \le \quad 2\sup|\chi| \quad &\le \quad 2,
\end{aligned}$$

we see that

$$\fint \chi\,dx \lesssim \frac{1}{R}\left(\fint ||\nabla|^{-1}(u-\Phi)|^2 dx \fint \chi dx\right)^{1/2} + E + RE.$$

Absorbing $\fint \chi\,dx$ on the right hand side after the application of Young's inequality yields

$$\fint \chi\,dx \lesssim \frac{1}{R}\fint ||\nabla|^{-1}(u-\Phi)|^2 dx + E + RE,$$

which after optimization in $R$ reads

$$\fint \chi\,dx \lesssim \left(\fint ||\nabla|^{-1}(u-\Phi)|^2 dx\right)^{1/3} E^{2/3} + E.$$

We can now plug this into estimate (9) and get

$$\begin{aligned}
\Phi &\le E^{1/2} + |\{u \ge \frac{2}{3}\}| \\
&\le E^{1/2} + \fint \chi\,dx \\
&\lesssim \left(\fint ||\nabla|^{-1}(u-\Phi)|^2 dx\right)^{1/3} E^{2/3} + E^{1/2} + E.
\end{aligned}$$

Hence for $E \ll \Phi^2 \leq 1$

$$\Phi \lesssim \left( \fint ||\nabla|^{-1}(u-\Phi)|^2 dx \right)^{1/3} E^{2/3},$$

that is

$$E \gtrsim \Phi^{3/2} \left( \fint ||\nabla|^{-1}(u-\Phi)|^2 dx \right)^{-1/2},$$

as was claimed. □

### 3.5 Interpolation inequalities and lower energy bounds ($d = 2$)

In two dimensions, the scaling of above estimates in $\Phi$ does not match the heuristic predictions of Section 3.2. Following Conti, Niethammer & Otto [CNO05] we can improve the interpolation lemma to achieve optimal scaling.

**Lemma 2.** *There is a constant $C \leq \infty$ such that for all $u : (0, \Lambda)^2 \to \mathbb{R}$, periodic with $\fint u\, dx = \Phi$ and all $s \geq 2\Phi$ we have*

$$\ln^{1/4} \frac{s}{\Phi}\, s|\{|u| \geq s\}|^{3/4} \leq C \|\nabla u\|_{L^1}^{1/2} \||\nabla|^{-1}(u-\Phi)\|_{L^2}^{1/2}.$$

Applying Lemma 2 to our $u$ for $s = 1$, we get

$$\ln^{1/4} \frac{1}{\Phi}\, |\{|u| \geq 1\}|^{3/4} \leq C \left( \int |\nabla u| dx \right)^{1/2} \left( \int \left| |\nabla|^{-1}(u-\Phi) \right|^2 \right)^{1/4},$$

that is

$$\Phi^{3/4} \ln^{1/4} \frac{1}{\Phi} = \ln^{1/4} \frac{1}{\Phi} \fint\{|u| \geq 1\}|^{3/4} \leq C \left( \fint |\nabla u| dx \right)^{1/2} \left( \fint \left| |\nabla|^{-1}(u-\Phi) \right|^2 \right)^{1/4}.$$

Hence

$$\fint |\nabla u| dx \geq \frac{1}{C} \Phi^{3/2} \ln^{1/2} \frac{1}{\Phi} \left( \fint \left| |\nabla|^{-1}(u-\Phi) \right|^2 \right)^{-1/2}.$$

Now Proposition 1 with $\alpha = 1$ and $E_0 = \Phi^{3/2} \ln^{1/2} \frac{1}{\Phi}$ gives

$$E \gtrsim \left( \Phi^{3/2} \ln^{1/2} \frac{1}{\Phi} \right)^{\frac{2}{1+2}} t^{-\frac{1}{1+2}} = \Phi \ln^{1/3} \frac{1}{\Phi} t^{-1/3}.$$

This scaling matches the heuristics and thus is optimal.

*Proof of Lemma 2.* The strategy of the proof is similar to the one in higher dimension, but we need a more careful choice of convolution kernels.

We introduce for $L > R > 0$ two families of kernels

$$K_R(z) = \begin{cases} \frac{1}{\pi R^2} & \text{if } |z| \leq R, \\ 0 & \text{if } |z| > R \end{cases}$$

and

$$K_{R,L}(z) = \begin{cases} \frac{1}{\pi R^2} & \text{if } |z| \leq R, \\ \frac{1}{\pi R^2} \frac{\ln \frac{L}{|z|}}{\ln \frac{L}{R}} & \text{if } R < |z| \leq L, \\ 0 & \text{if } |z| > L. \end{cases}$$

Note that $0 \leq K_R \leq K_{R,L}$ and $\int K_R = 1$.
Consider

$$\chi = \begin{cases} 1 & \text{where } u \geq s, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$s \int \chi \, dx \leq \int u\chi \, dx = \int u \min\{K_{R,L} * \chi, 1\} dx + \int u(\chi - \min\{K_{R,L} * \chi, 1\}) dx.$$

As $u \geq 0$, $K_{R,L} \geq K_R$, $\chi \geq 0$, this can be estimated by

$$\leq \int u \min\{K_{R,L} * \chi, 1\} dx + \int u(\chi - \min\{K_R * \chi, 1\}) dx,$$

and since $K_R * \chi \leq 1$ this is

$$= \int u \min\{K_{R,L} * \chi, 1\} dx + \int u(\chi - K_R * \chi) dx.$$

Splitting the first term and moving the convolution in the second we get

$$s \int \chi \, dx \leq \Phi \int \min\{K_{R,L} * \chi, 1\} dx + \int (u - \Phi) \min\{K_{R,L} * \chi, 1\} dx + \int \chi(u - K_R * u) dx.$$

Enlarging by dropping the first minimum and using $\chi \in [0,1]$, this can be bounded by

$$\leq \Phi \int K_{R,L} dx \int \chi \, dx + \left( \int \left| |\nabla|^{-1}(u - \Phi) \right|^2 dx \int |\nabla \min\{K_{R,L} * \chi, 1\}|^2 dx \right)^{1/2} + \int |u - K_R * u| dx$$

We consider the terms individually and see that

$$\int K_{R,L} dx \leq \frac{1}{\pi R^2} |B_L| = \left(\frac{L}{R}\right)^2,$$

$$\int |u - K_R * u| dx \lesssim R \int |\nabla u| dx,$$

and

$$\int |\nabla \min\{K_{R,L} * \chi, 1\}|^2 dx = \nabla(K_{R,L} * \chi) \cdot \nabla \min\{K_{R,L} * \chi, 1\} dx$$
$$= \int (-\Delta K_{R,L}) * \chi \min\{K_{R,L} * \chi, 1\} dx$$
$$\leq \int (-\Delta K_{R,L})_+ * \chi \, dx$$
$$= \int (-\Delta K_{R,L})_+ dx \int \chi \, dx$$
$$= \frac{2}{R^2} \frac{1}{\ln \frac{L}{R}} \int \chi \, dx.$$

Hence

$$s \int \chi \, dx \leq \Phi \left(\frac{L}{R}\right)^2 \int \chi \, dx + \left( \int \left||\nabla|^{-1}(u - \Phi)\right|^2 dx \frac{2}{R^2} \frac{1}{\ln \frac{L}{R}} \int \chi \, dx \right)^{1/2} + R \int |\nabla u| dx.$$

We choose $\frac{L}{R}$ such that

$$\Phi \left(\frac{L}{R}\right)^2 = \frac{1}{2} s, \quad \text{i.e.} \quad \frac{L}{R} = \left(\frac{1}{2} \frac{s}{\Phi}\right)^{1/2} \gg 1$$

to get

$$s \int \chi \, dx \lesssim \left( \int \left||\nabla|^{-1}(u - \Phi)\right|^2 dx \frac{1}{R^2} \frac{1}{\ln \frac{s}{\Phi}} \int \chi \, dx \right)^{1/2} + R \int |\nabla u| dx,$$

and absorbing $s \int \chi \, dx$ with Young's inequality

$$s \int \chi \, dx \lesssim \frac{1}{R^2} \frac{1}{s \ln \frac{s}{\Phi}} \int \left||\nabla|^{-1}(u - \Phi)\right|^2 dx + R \int |\nabla u| dx.$$

Finally we optimize in $R$ and obtain

$$s \int \chi \, dx \lesssim \left( \int |\nabla u| dx \right)^{2/3} \left( \frac{1}{s \ln \frac{s}{\Phi}} \int \left||\nabla|^{-1}(u - \Phi)\right|^2 dx \right)^{1/3},$$

that is

$$s^{4/3} \ln^{1/3} \frac{s}{\Phi} \int \chi \, dx \lesssim \left( \int |\nabla u| dx \right)^{2/3} \left( \int \left||\nabla|^{-1}(u - \Phi)\right|^2 dx \right)^{1/3},$$

which is the desired estimate

$$s \ln^{1/4} \frac{s}{\Phi} \left( \int \chi \, dx \right)^{3/4} \lesssim \left( \int |\nabla u| dx \right)^{1/2} \left( \int \left||\nabla|^{-1}(u - \Phi)\right|^2 dx \right)^{1/4}. \qquad \square$$

With the same strategy as in the case of higher dimensions, we can adapt our calculation to the actual Cahn-Hilliard energy.

**Proposition 3.** *There exists a constant* $C \leq \infty$ *such that for all* $\Lambda > 0$ *and* $u : (0,\Lambda)^2 \to \mathbb{R}$ *with* $\Phi = \fint u dx \ll 1$ *satisfying*

$$E := \fint \frac{1}{2}|\nabla u|^2 + \frac{1}{2}(u(1-u))^2 dx \leq \frac{1}{C}\Phi^2$$

*we have*

$$E \geq \frac{1}{C}\Phi^{3/2} \ln^{1/2} \frac{1}{\Phi} \left( \fint \left| |\nabla|^{-1}(u-\Phi) \right|^2 \right)^{-1/2}.$$

*Proof.* We again use for $L > R > 0$ the kernel families $K_R$ and $K_{R,L}$ of Lemma 2.

Consider as before

$$\chi = \begin{cases} 1 & \frac{2}{3} \leq u, \\ 3u - 1 & \frac{1}{3} \leq u \leq \frac{2}{3}, \\ 0 & u \leq \frac{1}{3}. \end{cases}$$

We proceed in a fashion similar to the proofs of Lemma 2 and Proposition 2, the only difference is that we cannot pass from $K_{R,L}$ to $K_R$ where $u < 0$. We estimate

$$\begin{aligned} \fint \chi \, dx \lesssim \fint u\chi \, dx &= \fint u \min\{K_{R,L} * \chi, 1\} dx + \int u(\chi - \min\{K_{R,L} * \chi, 1\}) dx \\ &\leq \fint \Phi \min\{K_{R,L} * \chi, 1\} dx + \fint (u - \Phi) \min\{K_{R,L} * \chi, 1\} dx \\ &\quad + 2\fint_{\{u \leq -1\} \cup \{u \geq 2\}} |u| dx + \fint_{\{0 \leq u \leq 2\}} u(\chi - K_R * \chi) dx \\ &\quad + \fint_{\{-1 \leq u \leq 0\}} u(\chi - \min\{K_{R,L} * \chi, 1\}) dx. \end{aligned}$$

Recall that we write $\fint$ to denote $\Lambda^{-2} \int$ (regardless of volume of the integration domain).

Our previous estimates in Proposition 2 and Lemma 2 readily apply to all but the last term. Noting $\chi = 0$ where $u \leq 0$, assuming $E \leq \Phi^2$ and with

$$\begin{aligned} \fint_{\{-1 \leq u \leq 0\}} -u \min\{K_{R,L} * \chi, 1\} dx &\lesssim \left( \fint_{-1 \leq u \leq 0} u^2 dx \right)^{1/2} \left( \int K_{R,L} dx \fint \chi \, dx \right)^{1/2} \\ &\lesssim E^{1/2} \frac{L}{R} \left( \fint \chi \, dx \right)^{1/2} \lesssim E^{1/2} + \Phi \left( \frac{L}{R} \right)^2 \fint \chi \, dx \end{aligned}$$

we see that

$$\fint \chi \, dx \lesssim \Phi \left( \frac{L}{R} \right)^2 \fint \chi \, dx + \left( \fint \left| |\nabla|^{-1}(u-\Phi) \right|^2 dx \frac{1}{R^2} \frac{1}{\ln \frac{L}{R}} \fint \chi \, dx \right)^{1/2} + E + RE + E^{1/2}.$$

As before we choose $L = C^{-1}\Phi^{-1/2} R$ and use Young's inequality to absorb all occurrences of $\fint \chi \, dx$ on the right hand side. After optimization in $R$ we get

$$\fint \chi \, dx \lesssim \ln^{-1/3} \frac{1}{\Phi} \left( \fint \left| |\nabla|^{-1}(u-\Phi) \right|^2 dx \right)^{1/3} E^{2/3} + E + E^{1/2}.$$

With this we follow the proof of Proposition 2 to calculate for $E \ll \Phi^2 \leq 1$

$$\Phi \lesssim E^{1/2} + \fint \chi dx \lesssim \ln^{-1/3} \frac{1}{\Phi} \left( \fint \left| |\nabla|^{-1} (u - \Phi) \right|^2 dx \right)^{1/3} E^{2/3} + E + E^{1/2}$$
$$\lesssim \ln^{-1/3} \frac{1}{\Phi} \left( \fint \left| |\nabla|^{-1} (u - \Phi) \right|^2 dx \right)^{1/3} E^{2/3},$$

that is

$$E \gtrsim \Phi^{3/2} \ln^{1/2} \frac{1}{\Phi} \left( \fint \left| |\nabla|^{-1} (u - \Phi) \right|^2 dx \right)^{-1/2},$$

as claimed. □

# 4 Energy functionals and scaling

In this section we prepare the setting for the proofs of Theorems 1 and 2 by rewriting the energy functionals and providing the anisotropic rescaling of the coordinates.

**Localizing the stray field.** First, we reformulate the magnetostatic energy to include $h$ in the minimization in order to make the problem more local. Recall that we defined the field energy as the squared $L^2$-norm of the $(-l, l)^2$-periodic field $h : \mathbb{R}^3 \to \mathbb{R}^3$ given by the simplified Maxwell equations

$$\begin{aligned} \nabla \cdot (h + m) &= 0 \text{ and} \\ \nabla \times h &= 0. \end{aligned}$$

As discussed with a bit more context in the appendix the energy may equivalently be expressed as the minimization

$$\int_{(-l,l)^2 \times \mathbb{R}} |h|^2 dx = \min \left\{ \int_{(-l,l)^2 \times \mathbb{R}} |\tilde{h}|^2 dx \,\middle|\, \tilde{h} : \mathbb{R}^3 \to \mathbb{R}^3 \text{ is } (-l,l)^2\text{-periodic in } x', \right. \\ \left. \nabla \cdot (\tilde{h} + m) = 0 \right\}.$$

Here and in the following the differential equations as in the last condition are understood in the sense of distributions. We denote by $x'$ the variables $\binom{x_1}{x_2}$ and do similarly for vector fields and derivatives. Hence, setting

$$e_{Q,d,t,l}(m,h) := \frac{1}{4l^2} \left( d^2 \int_{(-l,l)^2 \times (-t,t)} |\nabla m|^2 dx + Q \int_{(-l,l)^2 \times (-t,t)} |m'|^2 dx + \int_{(-l,l)^2 \times \mathbb{R}} |h|^2 dx \right)$$

we have

$$e(Q,d,t,l) = \min \left\{ e_{Q,d,t,l}(m,h) \,\middle|\, m, h : \mathbb{R}^3 \to \mathbb{R}^3 \text{ are } (-l,l)^2\text{-periodic in } x', \right. \\ |m|^2 = \begin{cases} 1 & \text{for } x_3 \in (-t,t), \\ 0 & \text{otherwise,} \end{cases} \\ \left. \nabla \cdot (h + m) = 0 \right\}.$$

**Rescaling.** We now rescale the lengths, fields and energy to more convenient units. The horizontal lengths are scaled so that the width of bulk domains is of order 1, i.e.

$$x' = (dQ^{1/2})^{1/3} t^{2/3} \hat{x}' \qquad \text{and} \qquad l = (dQ^{1/2})^{1/3} t^{2/3} \hat{l},$$

the vertical length is normalized so that the sample is on the interval $(-1,1)$ in $\hat{x}_3$-direction, i.e.

$$x_3 = t\hat{x}_3.$$

In order to retain the structure of the magnetic field, we have to rescale the horizontal field according to

$$h' = \frac{(dQ^{1/2})^{1/3} t^{2/3}}{t} \hat{h}' = \left(\frac{dQ^{1/2}}{t}\right)^{1/3} \hat{h}'$$

and keep the vertical component $h_3 = \hat{h}_3$ to ensure $\nabla \cdot h = \frac{1}{t} \hat{\nabla} \cdot \hat{h}$. For consistency we write $m = \hat{m}$. With these rescalings in the coordinates it is convenient to also rescale the energy density as

$$e = (dQ^{1/2})^{2/3} t^{1/3} \hat{e}$$

in order to non-dimensionalize the weights in the energy. We write the energy in the new coordinates and quantities

$$\begin{aligned} e = (dQ^{1/2})^{2/3} t^{1/3} \hat{e} = \frac{t}{4\hat{l}^2} \Bigg( d^2 \int_{(-\hat{l},\hat{l})\times(-1,1)} & \left| \begin{pmatrix} \frac{1}{(dQ^{1/2})^{1/3} t^{2/3}} \hat{\nabla}' \\ \frac{1}{t} \frac{\partial}{\partial \hat{x}_3} \end{pmatrix} \hat{m} \right|^2 d\hat{x} \\ & + Q \int_{(-\hat{l},\hat{l})\times(-1,1)} |\hat{m}'|^2 d\hat{x} + \int_{(-\hat{l},\hat{l})^2\times\mathbb{R}} \left| \begin{pmatrix} \frac{(dQ^{1/2})^{1/3} t^{2/3}}{t} \hat{h}' \\ \hat{h}_3 \end{pmatrix} \right|^2 d\hat{x} \Bigg), \end{aligned}$$

and thus

$$\begin{aligned} \hat{e} &= \frac{1}{4\hat{l}^2} \Bigg( \left(\frac{d}{tQ}\right)^{2/3} \int_{(-\hat{l},\hat{l})^2\times(-1,1)} \left| \begin{pmatrix} \hat{\nabla}' \\ \left(\frac{dQ^{1/2}}{t}\right)^{1/3} \frac{\partial}{\partial \hat{x}_3} \end{pmatrix} \hat{m} \right|^2 d\hat{x} \\ &\qquad + \left(\frac{tQ}{d}\right)^{2/3} \int_{(-\hat{l},\hat{l})\times(-1,1)} |\hat{m}'|^2 d\hat{x} + \int_{(-\hat{l},\hat{l})^2\times\mathbb{R}} \left| \begin{pmatrix} \hat{h}' \\ \left(\frac{t}{dQ^{1/2}}\right)^{1/3} \hat{h}_3 \end{pmatrix} \right|^2 d\hat{x} \Bigg) \\ &= \frac{1}{4\hat{l}^2} \left( \delta \int_{(-\hat{l},\hat{l})^2\times(-1,1)} \left| \begin{pmatrix} \hat{\nabla}' \\ \varepsilon \frac{\partial}{\partial \hat{x}_3} \end{pmatrix} \hat{m} \right|^2 d\hat{x} + \frac{1}{\delta} \int_{(-\hat{l},\hat{l})\times(-1,1)} |\hat{m}'|^2 d\hat{x} + \int_{(-\hat{l},\hat{l})^2\times\mathbb{R}} \left| \begin{pmatrix} \hat{h}' \\ \frac{1}{\varepsilon} \hat{h}_3 \end{pmatrix} \right|^2 d\hat{x} \right) \\ &=: \hat{e}_{\delta,\varepsilon,\hat{l}}(\hat{m}, \hat{h}) \end{aligned}$$

when we set

$$\delta := \left(\frac{d}{tQ}\right)^{2/3} = \frac{d/Q^{1/2}}{(dQ^{1/2})^{1/3} t^{2/3}} = \frac{\text{Bloch wall width}}{\text{bulk domain width}}$$

and

$$\varepsilon := \left(\frac{dQ^{1/2}}{t}\right)^{1/3} = \frac{(dQ^{1/2})^{1/3} t^{2/3}}{t} = \frac{\text{bulk domain width}}{\text{sample thickness}}.$$

The rescaling of $h'$ now reads $h' = \varepsilon \hat{h}'$ and the constraints turn into

$$|\hat{m}|^2 = \begin{cases} 1 & \text{for } \hat{x}_3 \in (-1,1), \\ 0 & \text{otherwise} \end{cases}$$

and

$$\hat{\nabla}' \cdot (\hat{h} + \frac{1}{\varepsilon}\hat{m}') + \frac{\partial}{\partial \hat{x}_3}(\hat{h}_3 + \hat{m}_3) = 0.$$

Finally, observe that we can now conveniently characterize the parameter regime of interest because $Q \gg 1$, $dQ^{1/2} \ll t$, and $(dQ^{1/2})^{1/3}t^{2/3} \ll l$ are equivalent to $\delta \ll \varepsilon^2$, $\varepsilon^2 \ll 1$, and $1 \ll \hat{l}$, respectively. Combining, we are interested in

$$\delta \ll \varepsilon^2 \ll 1 \ll \hat{l}.$$

Hence, we define

$$\hat{e}(\delta, \varepsilon, \hat{l}) := \min \big\{ \hat{e}_{\delta,\varepsilon,\hat{l}}(\hat{m}, \hat{h}) \big| \hat{m}, \hat{h} : \mathbb{R}^3 \to \mathbb{R}^3, \quad (-\hat{l}, \hat{l})^2\text{-periodic}, \quad |\hat{m}|^2 = \begin{cases} 1 & \text{if } \hat{x}_3 \in (-1,1), \\ 0 & \text{otherwise}, \end{cases} \\ \hat{\nabla}' \cdot (\hat{h}' + \frac{1}{\varepsilon}\hat{m}') + \frac{\partial}{\partial \hat{x}_3}(\hat{h}_3 + \hat{m}_3) = 0 \big\}$$

and to prove Theorem 2 we have to show that

$$\lim_{\delta/\varepsilon^2 \to 0, \varepsilon \to 0, \hat{l} \to \infty} \hat{e}(\delta, \varepsilon, \hat{l}) \in (0, \infty),$$

i.e. that the limit exists and is a strictly positive real number. Because we frequently have to take simultaneous limits we introduce the notation

$$\lim_{\delta \ll \varepsilon^2 \ll 1 \ll \hat{l}} \hat{e}(\delta, \varepsilon, \hat{l}) := \lim_{\delta/\varepsilon^2 \to 0, \varepsilon \to 0, 1/\hat{l} \to 0} \hat{e}(\delta, \varepsilon, \hat{l}),$$

where, as with usual limits of real variables, we say that the limit exists if all sequences of (positive) parameters satisfying the limiting relations, i.e. such that all quotients of the left hand side and right hand side of »$\ll$« converge to 0, have a limit that is independent of the particular choice of the sequence.

Having adequately reformulated the problem, we proceed from this point using the rescaled quantities exclusively and drop all »ˆ«.

We now discuss the various energy functionals and minimization problems that are useful in the following analysis. Our starting point is the energy

$$e_{\delta,\varepsilon,l}(m, h) = \frac{1}{4l^2} \left( \delta \int_{(-l,l)^2 \times (-1,1)} \left| \begin{pmatrix} \nabla' \\ \varepsilon \partial_3 \end{pmatrix} m \right|^2 dx + \frac{1}{\delta} \int_{(-l,l)^2 \times (-1,1)} |m'|^2 \, dx + \int_{(-l,l)^2 \times \mathbb{R}} \left| \begin{pmatrix} h' \\ \frac{1}{\varepsilon} h_3 \end{pmatrix} \right|^2 dx \right),$$

and its minimum on the periodic configurations

$$e^p(\delta, \varepsilon, l) = \min \Big\{ e_{\delta,\varepsilon,l}(m, h) \, \Big| \, m, h : \mathbb{R}^3 \to \mathbb{R}^3, \quad (-l, l)^2\text{-periodic}, \quad |m|^2 = \begin{cases} 1 & \text{if } x_3 \in (-1,1), \\ 0 & \text{otherwise}, \end{cases} \\ \nabla' \cdot (h' + \frac{1}{\varepsilon} m') + \partial_3(h_3 + m_3) = 0 \Big\}.$$

We also introduce the renormalized, sharp interface energy where $m'$ and $h_3$ have vanished, the exchange and anisotropy terms have been replaced by a $BV$-norm, and $\partial_3 m_3$ has ceased to play a role. The renormalized energy functional is

$$e_l(m_3, h') = \frac{1}{4l^2} \left( 2 \int_{[-l,l)^2 \times (-1,1)} |\nabla' m_3| \, dx + \int_{(-l,l)^2 \times \mathbb{R}} |h'|^2 \, dx \right)$$

and we consider the corresponding minimization problem amongst periodic $m_3, h'$, i.e. we are interested in

$$e^p(l) = \min\Big\{e_l(m_3,h')\Big|\, m_3:\mathbb{R}^3\to\mathbb{R}, h':\mathbb{R}^3\to\mathbb{R}^2,\quad (-l,l)^2\text{-periodic},\quad m_3^2=\begin{cases}1 & \text{if } x_3\in(-1,1),\\ 0 & \text{otherwise,}\end{cases}$$
$$\nabla'\cdot h' + \partial_3 m_3 = 0\Big\}.$$

The first term in the energy is understood in the sense of $BV$-functions periodic in $x'$, i.e.

$$\int_{[-l,l)^2\times(-1,1)} |\nabla' m_3|dx$$
$$= \sup\Big\{\int_{(-l,l)^2\times(-1,1)} m_3\,\nabla'\cdot\xi' dx\,\Big|\,\xi'\in C^\infty(\mathbb{R}^2\times\mathbb{R}),\ (-l,l)^2\text{-periodic in } x',$$
$$|\xi'|\le 1 \text{ in } (-l,l)^2\times(-1,1)\Big\}.$$

Although the half-open fundamental cell is not used on the right hand side, we use it to be consistent with the interpretation of the integral as the measure $|\nabla' m_3|$ of the domain of integration.

We remark that the reduced energy enforces that $m_3$ vanishes weakly at the top and bottom sample boundary because finiteness of the field term excludes jumps of $m_3$ in the $x_3$-direction. But then the field vanishes outside the sample domain. Thus one could also prescribe $m_3$ to vanish weakly at top and bottom as boundary conditions and only take integrals over the sample domain. This insight is crucial to justify Theorem 3 and used in its proof in Section 8.

As $m_3\in\{\pm1\}$ almost everywhere in $(-l,l)^2\times(-1,1)$ we have an interpretation of the reduced $BV$-gradient as a "slicewise measure" of the interface that corresponds to the usual geometric interpretation of the total gradient of a characteristic function of a set as the perimeter. To be precise,

$$\int_{[-l,l)^2\times(-1,1)} |\nabla' m_3|dx = 2\int_{-1}^{1}\mathscr{H}^1(\partial\{m_3(\,\cdot\,,x_3)=1\})dx_3.$$

Note, however, that to make this interpretation rigorous one needs to show that the distribution used above to define the left hand side indeed is represented by a measure that is absolutely continuous with respect to the Lebesgue measure $dx_3$. For energy minimizers, the argument in the proof of [KM94, Lemma 2.6] is also valid in the present setting. We do not use this interpretation except in constructions where the absolute continuity is evident.

Loosely speaking, one key message of Theorem 2 is the following: As the system size $l$ becomes large w.r.t. the intrinsic length scale of the pattern of Figure 2, boundary conditions become irrelevant. This phenomenon is reminiscent of Gibbs states in Ising models below the critical temperature. In fact, we shall use a similar analysis tool by working with different boundary conditions: Next to the periodic boundary conditions, we shall use free boundary conditions, i.e. consider

$$e^f(\delta,\varepsilon,l) = \min\Big\{e_{\delta,\varepsilon,l}(m,h)\,\Big|\, m,h:(-l,l)^2\times\mathbb{R}\to\mathbb{R}^3,\quad |m|^2=\begin{cases}1 & \text{if } x_3\in(-1,1),\\ 0 & \text{otherwise,}\end{cases}$$
$$\nabla'\cdot(h'+\frac{1}{\varepsilon}m') + \partial_3(h_3+m_3) = 0\Big\},$$

and

$$e^f(l) = \min\Big\{e_l(m_3,h')\,\Big|\, m_3 : (-l,l)^2\times\mathbb{R}\to\mathbb{R}, h' : (-l,l)^2\times\mathbb{R}\to\mathbb{R}^2,$$
$$m_3^2 = \begin{cases}1 & \text{if } x_3\in(-1,1),\\ 0 & \text{otherwise,}\end{cases} \qquad \nabla'\cdot h' + \partial_3 m_3 = 0\Big\}.$$

As discussed for the full energy, the role of $h$ or $h'$ is to allow a local formulation of the stray field energy. Equivalently, we could write the corresponding terms as negative norms, characterized either by a Fourier-multiplier or via solving an auxiliary problem, and the energies as only depending on $m$ or $m_3$. Inspired by these interpretations of $h$ and $h'$ we introduce the formal notation

$$\int_{(-l,l)^2\times\mathbb{R}} \big||\nabla'|^{-1}\partial_3 m_3\big|^2\,dx = \min\Big\{\int_{(-l,l)^2\times\mathbb{R}}|h'|^2dx\Big| h' : \mathbb{R}^2\times\mathbb{R}\to\mathbb{R}^2 \text{ is } (-l,l)^2\text{-periodic in } x',$$
$$\nabla'\cdot h' + \partial_3 m_3 = 0 \text{ in } \mathbb{R}^3\Big\},$$

and

$$\int_{(-l,l)^2\times\mathbb{R}} \Big|(|\nabla'|^2+\varepsilon^2|\partial_3|^2)^{-1/2}\partial_3 m_3\Big|^2 dx$$
$$= \min\Big\{\int_{(-l,l)^2\times\mathbb{R}}\Big|\begin{pmatrix}h'\\ \frac{1}{\varepsilon}h_3\end{pmatrix}\Big|^2dx\,\Big|\, h : \mathbb{R}^2\times\mathbb{R}\to\mathbb{R}^3 \text{ is } (-l,l)^2\text{-periodic in } x',$$
$$\nabla\cdot h + \partial_3 m_3 = 0 \text{ in } \mathbb{R}^3\Big\}. \tag{10}$$

To conclude this section let us briefly consider the scaling behavior of the sharp interface energy. We start with the energy (not divided by the cross-section area) on a $x'$-periodic domain $(-l,l)^2\times(-t,t)$

$$E_{l,t}(m_3,h') = 2\int_{[-l,l)^2\times(-t,t)}|\nabla' m_3|dx + \int_{(-l,l)^2\times\mathbb{R}}|h'|^2dx$$

for configurations

$$m_3 : (-l,l)^2\times\mathbb{R}\to\mathbb{R}, \qquad h' : (-l,l)^2\times\mathbb{R}\to\mathbb{R}^2$$

with

$$m_3^2 = \begin{cases}1 & \text{if } x_3\in(-1,1),\\ 0 & \text{otherwise}\end{cases}$$

and

$$\nabla'\cdot h' + \partial_3 m_3 = 0. \tag{11}$$

Scaling the coordinate in the magnetization $\tilde m_3(s_{x'}x', s_{x_3}x_3) = m_3(x',x_3)$ and the field as

$$\tilde h'(s_{x'}x', s_{x_3}x_3) = \frac{s_{x'}}{s_{x_3}}h'(x',x_3)$$

preserves (11). Then

$$E_{s_{x'}l,s_{x_3}t}(\tilde m_3,\tilde h') = 2s_{x'}^2 s_{x_3}\int_{(-l,l)^2\times(-t,t)} s_{x'}^{-1}|\nabla' m_3|dx + s_{x'}^2 s_{x_3}\int_{(-l,l)^2\times\mathbb{R}}\left(\frac{s_{x'}}{s_{x_3}}\right)^2|h'|^2dx.$$

After the equilibrating choice $s_{x_3}=s$, $s_{x'}=s^{2/3}$ this becomes

$$E_{s^{2/3}l,st}(\tilde m_3,\tilde h') = s^{5/3}E_{l,t}(m_3,h').$$

# 5 Convergence to the sharp interface model

This section is devoted to the proof of the almost-Γ-limit Theorem 1. We first restate it in the rescaled coordinates.

**Theorem 4** (Theorem 1 made precise)**.** *For fixed length $l$, the reduced energy is an upper and lower $\Gamma$-type limit of the full energy for*

$$\delta/\varepsilon^2 \to 0 \qquad \text{and} \qquad \varepsilon^2 \to 0 \tag{12}$$

*in the following sense.*

1. *The energy of any pair $(m_3, h')$ admissible in $e^p(l)$ can be approximated in the regime (12) by the energy of pairs $(m^{(\varepsilon,\delta)}, h^{(\varepsilon,\delta)})$ admissible for $e^p(\varepsilon,\delta,l)$ such that*

$$\lim_{\delta \ll \varepsilon^2 \ll 1} e_{\delta,\varepsilon,l}(m^{(\varepsilon,\delta)}, h^{(\varepsilon,\delta)}) \leq e_l(m_3, h').$$

2. *If $\delta^{(\nu)}, \varepsilon^{(\nu)}$ converge as in (12) and $(m^{(\nu)}, h^{(\nu)})$ is admissible for $e^f(\delta^{(\nu)}, \varepsilon^{(\nu)}, l)$ with*

$$m_3^{(\nu)} \stackrel{w*}{\rightharpoonup} m_3 \text{ in } L^\infty((-l,l)^2 \times \mathbb{R}) \qquad \text{and} \qquad h^{(\nu)\prime} \stackrel{w}{\rightharpoonup} h' \text{ in } L^2((-l,l)^2 \times \mathbb{R})$$

*then $(m_3, h')$ is admissible for $e^f(l)$ and*

$$e_l(m_3, h') \leq \liminf_{\nu \uparrow \infty} e_{\delta^{(\nu)},\varepsilon^{(\nu)},l}(m^{(\nu)}, h^{(\nu)}).$$

Note that the theorem does not provide a Γ-limit result because the lower bound and the approximation are done in regimes with different boundary conditions and we do not actually verify the approximation property of our prospective recovery sequence. This could be fixed, but as our main interest is Theorem 2, we omit stating and proving a theorem concerning a proper Γ-limit.

We start with the more straightforward lower bound, which is demonstrated by a compensated compactness argument that takes into account the anisotropy. Then we address the upper bound which requires a more involved proof.

We thus begin with the proof of Theorem 4, part 2. This is also used to show

$$\liminf_{\delta \ll \varepsilon^2 \ll 1} e^f(\delta, \varepsilon, l) \geq e^f(l).$$

for the proof of Theorem 2.

*Proof of Theorem 4, part 2.* We fix $l$ and recall that we aim to show the following: Given any sequences $\{\delta^{(\nu)}, \varepsilon^{(\nu)}\} \subset (0,\infty)$ let $(m^{(\nu)}, h^{(\nu)})$ be admissible for $e^f(\delta^{(\nu)}, \varepsilon^{(\nu)}, l)$ with

$$\begin{aligned} \delta^{(\nu)} \to 0, \quad \varepsilon^{(\nu)} \to 0, \quad \frac{\delta^{(\nu)}}{(\varepsilon^{(\nu)})^2} \to 0, \\ m_3^{(\nu)} \stackrel{w*}{\rightharpoonup} m_3 \text{ in } L^\infty((-l,l)^2 \times \mathbb{R}), \end{aligned} \tag{13}$$

and

$$h^{(\nu)\prime} \stackrel{w}{\rightharpoonup} h' \text{ in } L^2((-l,l)^2 \times \mathbb{R})$$

as $\nu \to \infty$, then

$$\begin{aligned}&(m_3, h') \text{ is admissible for } e^f(l) \text{ and}\\ &e_l(m_3, h') \le \liminf_{\nu\uparrow\infty} e_{\delta^{(\nu)},\varepsilon^{(\nu)},l}(m^{(\nu)}, h^{(\nu)}).\end{aligned}$$

Without compromising generality, we may assume that the energy $e_{\delta^{(\nu)},\varepsilon^{(\nu)},l}(m^{(\nu)}, h^{(\nu)})$ remains bounded. In particular, this implies that

$$\frac{1}{\delta^{(\nu)}}\int_{(-l,l)^2\times(-1,1)} |m^{(\nu)\prime}|^2 dx \qquad \text{and} \qquad \int_{(-l,l)^2\times\mathbb{R}} \left|\frac{1}{\varepsilon^{(\nu)}} h_3^{(\nu)}\right|^2 dx$$

are bounded and with $\frac{\delta^{(\nu)}}{(\varepsilon^{(\nu)})^2} \to 0$ we see

$$\int_{(-l,l)^2\times(-1,1)} \left|\frac{1}{\varepsilon^{(\nu)}} m^{(\nu)\prime}\right|^2 dx \to 0 \qquad \text{as well as} \qquad \int_{(-l,l)^2\times\mathbb{R}} |h_3^{(\nu)}|^2 dx \to 0. \tag{14}$$

Thus, the differential equation

$$\nabla' \cdot \big(h^{(\nu)\prime} + \frac{1}{\varepsilon^{(\nu)}} m^{(\nu)\prime}\big) + \partial_3(h_3^{(\nu)} + m_3^{(\nu)}) = 0$$

yields

$$\nabla' \cdot h' + \partial_3 m_3 = 0$$

in the limit as desired.

We first bound $4l^2 e_{\delta^{(\nu)},\varepsilon^{(\nu)},l}(m^{(\nu)}, h^{(\nu)})$ from below. In the sample the magnetization satisfies the pointwise estimate

$$\begin{aligned}
\delta^{(\nu)}\left|\begin{pmatrix}\nabla' \\ \varepsilon^{(\nu)}\partial_3\end{pmatrix} m^{(\nu)}\right|^2 + \frac{1}{\delta^{(\nu)}}\left|m^{(\nu)\prime}\right|^2 &\ge \delta^{(\nu)}\left|\nabla' m^{(\nu)}\right|^2 + \frac{1}{\delta^{(\nu)}}\left|m^{(\nu)\prime}\right|^2\\
&\ge \delta^{(\nu)}\left|\nabla' |m^{(\nu)}|\right|^2 + \frac{1}{\delta^{(\nu)}}\left|m^{(\nu)\prime}\right|^2\\
&= \delta^{(\nu)}\frac{(m_3^{(\nu)})^2}{1-(m_3^{(\nu)})^2}\left|\nabla' m_3^{(\nu)}\right|^2 + \frac{1}{\delta^{(\nu)}}\Big(1-(m_3^{(\nu)})^2\Big)\\
&\ge 2\left|\nabla' m_3^{(\nu)}\right|.
\end{aligned}$$

Dropping the third component in $h$ we see that the renormalized energy provides a lower bound

$$\begin{aligned}
&4l^2 e_{\delta^{(\nu)},\varepsilon^{(\nu)},l}(m^{(\nu)}, h^{(\nu)})\\
&\quad= \delta^{(\nu)}\int_{(-l,l)^2\times(-1,1)}\left|\begin{pmatrix}\nabla' \\ \varepsilon^{(\nu)}\partial_3\end{pmatrix} m^{(\nu)}\right|^2 dx + \frac{1}{\delta^{(\nu)}}\int_{(-l,l)^2\times(-1,1)}\left|m^{(\nu)\prime}\right|^2 dx\\
&\qquad+\int_{(-l,l)^2\times\mathbb{R}}\left|\begin{pmatrix} h^{(\nu)\prime} \\ \frac{1}{\varepsilon}h_3^{(\nu)}\end{pmatrix}\right|^2 dx\\
&\quad\ge 2\int_{(-l,l)^2\times(-1,1)}\left|\nabla' m_3^{(\nu)}\right| dx + \int_{(-l,l)^2\times\mathbb{R}}\left|h^{(\nu)\prime}\right|^2 dx.
\end{aligned}$$

By the lower semicontinuity of convex functionals under weak convergence this implies

$$\begin{aligned}\liminf_{\nu\uparrow\infty} 4l^2 e_{\delta^{(\nu)},\varepsilon^{(\nu)},l}(m^{(\nu)},h^{(\nu)}) &\geq 2\int_{(-l,l)^2\times(-1,1)} |\nabla' m_3|\,dx + \int_{(-l,l)^2\times\mathbb{R}} |h'|^2\,dx \\ &= 4l^2 e_l(m_3,h').\end{aligned}$$

It remains to show

$$m_3^2 = 1 \text{ a.e. in } (-l,l)^2\times(-1,1).$$

Since

$$|m^{(\nu)}|^2 = 1 \text{ if } x_3\in(-1,1),$$

this follows upon establishing

$$\begin{aligned}&m^{(\nu)\prime} \longrightarrow 0 \text{ in } L^2((-l,l)^2\times(-1,1)),\\ &m_3^{(\nu)} \longrightarrow m_3 \text{ pointwise a.e. in } (-l,l)^2\times(-1,1).\end{aligned}$$

In other words, we need compactness (only) for the nonconvex part. The first convergence follows readily from (14). The second is a consequence of (13) if we can show compactness of $\{m_3^{(\nu)}\}_{\nu\uparrow\infty}$ in form of

$$\int_{(-\tilde l,\tilde l)^2\times\mathbb{R}} |m_3^{(\nu)}(x'+h,x_3) - m_3^{(\nu)}(x',x_3)|dx \leq C|h| \text{ for } |h|\leq l-\tilde l, \tag{15}$$

$$\left(\int_{(-\tilde l,\tilde l)^2\times\mathbb{R}} |m_3^{(\nu)}(x',x_3+\tau) - m_3^{(\nu)}(x',x_3)|^2dx\right)^{1/2} \leq C|\tau|^{1/3}+o(1) \quad \text{for } |\tau|\leq (l-\tilde l)^{3/2}, \tag{16}$$

for $\tilde l \leq l$, that is, the modulus of continuity w.r.t. $x'$ and $x_3$ must decrease uniformly in $L^1$ and $L^2$, respectively, as $\nu\uparrow\infty$. Then, by the usual $L^p$-compactness criterion of M. Riesz (see e.g. [Ada75, Theorem 2.21]) the sequence is precompact in $L^1$. The other two requirements for the application of M. Riesz's criterion, uniform boundedness and that the norm taken on ever thinner boundary layers vanishes uniformly, are evident because $|m_3^{(\nu)}|\leq 1$ and $l$ is fixed. As we already know the limit of converging subsequences, $m_3^{(\nu)}\to m_3$ in $L^1$ and thus pointwise a.e., we are done upon establishing (15) and (16).

Inequality (15) is an immediate consequence of our bound on $e_l(m_3^{(\nu)},h^{(\nu)\prime})$ because for $|h|\leq l-\tilde l$

$$\begin{aligned}&\int_{(-\tilde l,\tilde l)^2\times\mathbb{R}} |m_3^{(\nu)}(x'+h,x_3) - m_3^{(\nu)}(x',x_3)|dx\\ &= \int_{(-\tilde l,\tilde l)^2\times(-1,1)} |m_3^{(\nu)}(x'+h,x_3) - m_3^{(\nu)}(x',x_3)|dx\\ &\leq |h|\int_{(-l,l)^2\times(-1,1)} |\nabla' m_3^{(\nu)}|dx \leq C|h|.\end{aligned}$$

For the second inequality, (16), we use a compensated compactness argument in the sense that we can combine the uniform modulus of continuity in $x'$ in a strong norm (cf. (15)) with the uniform modulus of continuity in $x_3$ in a weak (negative) norm provided by the field energy to obtain a uniform modulus of continuity in $x_3$ also in a strong norm (cf. (16)). We fix a smooth convolution kernel $\varphi:\mathbb{R}^2\to\mathbb{R}$ such that $\varphi \geq 0$, $\varphi(x') = 0$ for $|x'| \geq 1$, $\varphi(-x') = \varphi(x')$, and $\int_{\mathbb{R}^2} \varphi\, dx' = 1$ and denote by a subscript $_\alpha$ the convolution with $\frac{1}{\alpha^2}\varphi(\frac{\cdot}{\alpha})$. We observe that the equation

$$\partial_3(h_3^{(\nu)} + m_3^{(\nu)}) = -\nabla' \cdot (h^{(\nu)\prime} + \frac{1}{\varepsilon^{(\nu)}} m^{(\nu)\prime})$$

in $(-l,l)^2 \times \mathbb{R}$ implies for $\alpha \leq l - \tilde{l}$

$$\partial_3(h_3^{(\nu)} + m_3^{(\nu)})_\alpha = -\nabla' \cdot (h^{(\nu)\prime} + \frac{1}{\varepsilon^{(\nu)}} m^{(\nu)\prime})_\alpha$$

in $(-\tilde{l},\tilde{l})^2 \times \mathbb{R}$.

Hence

$$\begin{aligned}
&\left(\int_{(-\tilde{l},\tilde{l})^2\times\mathbb{R}} \left|\partial_3(h_3^{(\nu)} + m_3^{(\nu)})_\alpha\right|^2 dx\right)^{1/2} \\
&= \left(\int_{(-\tilde{l},\tilde{l})^2\times\mathbb{R}} \left|\nabla' \cdot (h^{(\nu)\prime} + \frac{1}{\varepsilon^{(\nu)}} m^{(\nu)\prime})_\alpha\right|^2 dx\right)^{1/2} \\
&\leq \int_{\mathbb{R}^2} \left|\frac{1}{\alpha^3}\nabla'\varphi\left(\frac{y'}{\alpha}\right)\right| dy' \left(\int_{(-l,l)^2\times\mathbb{R}} \left|h^{(\nu)\prime} + \frac{1}{\varepsilon^{(\nu)}} m^{(\nu)\prime}\right|^2 dx\right)^{1/2} \\
&\leq C\frac{1}{\alpha}\left(\left(\int_{(-l,l)^2\times\mathbb{R}} \left|h^{(\nu)\prime}\right|^2 dx\right)^{1/2} + \frac{\delta^{(\nu)}}{(\varepsilon^{(\nu)})^2}\frac{1}{\delta^{(\nu)}}\left(\int_{(-l,l)^2\times\mathbb{R}} \left|m^{(\nu)\prime}\right|^2 dx\right)^{1/2}\right) \\
&\leq C\frac{1}{\alpha} \cdot (C + o(1)\, C) \quad \leq \quad C\frac{1}{\alpha}.
\end{aligned}$$

As a consequence,

$$\left(\int_{(-\tilde{l},\tilde{l})^2\times\mathbb{R}} |(m_3^{(\nu)} + h_3^{(\nu)})_\alpha(x', x_3 + \tau) - (m_3^{(\nu)} + h_3^{(\nu)})_\alpha(x', x_3)|^2 dx\right)^{1/2} = C\frac{|\tau|}{\alpha}.$$

Finally, we observe that

$$\int_{(-l,l)^2\times\mathbb{R}} |h_3^{(\nu)}|^2 dx \leq (\varepsilon^{(\nu)})^2 \int_{(-l,l)^2\times\mathbb{R}} \left|\frac{1}{\varepsilon^{(\nu)}} h_3^{(\nu)}\right|^2 dx = o(1).$$

Combining these two estimates with inequality (15), we obtain for $\alpha \leq l - \tilde{l}$

$$\left(\int_{(-\tilde{l},\tilde{l})^2\times\mathbb{R}} |m_3^{(\nu)}(x', x_3 + \tau) - m_3^{(\nu)}(x', x_3)|^2 dx\right)^{1/2}$$

$$\leq \left(\int_{(-\tilde{l},\tilde{l})^2\times\mathbb{R}} |(m_3^{(\nu)} + h_3^{(\nu)})_\alpha(x', x_3 + \tau) - (m_3^{(\nu)} + h_3^{(\nu)})_\alpha(x', x_3)|^2 dx\right)^{1/2}$$

$$+ 2\left(\int_{(-\tilde{l},\tilde{l})^2\times\mathbb{R}} |(h_3^{(\nu)})_\alpha|^2 dx\right)^{1/2}$$

$$+ 2\left(\int_{(-\tilde{l},\tilde{l})^2\times\mathbb{R}} |(m_3^{(\nu)})_\alpha - m_3^{(\nu)}|^2 dx\right)^{1/2}$$

$$\leq C\frac{|\tau|}{\alpha} + 2\left(\int_{(-l,l)^2\times\mathbb{R}} |h_3^{(\nu)}|^2 dx\right)^{1/2} + 2^{3/2}\left(\int_{(-\tilde{l},\tilde{l})^2\times\mathbb{R}} |(m_3^{(\nu)})_\alpha - m_3^{(\nu)}| dx\right)^{1/2}$$

$$\leq C\frac{|\tau|}{\alpha} + o(1) + 2^{3/2}\left(\sup_{|k'|\leq\alpha}\int_{(-\tilde{l},\tilde{l})^2\times\mathbb{R}} |m_3^{(\nu)}(x' + k', x_3) - m_3^{(\nu)}(x', x_3)| dx\right)^{1/2}$$

$$\leq C\frac{|\tau|}{\alpha} + o(1) + C\alpha^{1/2}.$$

With the choice of $\alpha = |\tau|^{2/3}$ this is (16), and the above reasoning yields the desired $m_3^2 = 1$ for a.e. $x \in (-l, l)^2 \times (-1, 1)$.

Thus $(m_3, h')$ is admissible and the proof of our claim is complete. □

We now wish to prove Theorem 4, part 1, also needed to obtain

$$\limsup_{\delta \ll \varepsilon^2 \ll 1} e^p(\delta, \varepsilon, l) \leq e^p(l)$$

in the proof of Theorem 2.

The approximation is done in two steps. In the first, given by Lemma 3, we energetically approximate $(m_3, h')$ admissible for $e^p(l)$ by a pair $(m_3, h)$ for which we allow a small third field component but require some additional regularity of $m_3$ in the third direction. This can be seen as a counterpart to taking the limit of extreme anisotropy. In the second step we use Proposition 4 to "revert" the Modica-Mortola type passage from a diffuse to a sharp interface energy. We defer the proof of that proposition to Section 7.

More precisely we introduce an intermediate energy

$$e_{\varepsilon,l}(m_3, h) = (2l)^{-2}\left(2\int_{[-l,l)^2\times(-1,1)} \left|\begin{pmatrix}\nabla' \\ \varepsilon\partial_3\end{pmatrix} m_3\right| dx + \int_{(-l,l)^2\times\mathbb{R}} \left|\begin{pmatrix} h' \\ \frac{1}{\varepsilon}h_3\end{pmatrix}\right|^2 dx\right)$$

for pairs $(m_3, h)$ satisfying

$$m_3 : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}, h : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^3 \text{ are } (-l, l)^2\text{-periodic in } x',$$
$$m_3^2 = \begin{cases} 1 & \text{if } x_3 \in (-1, 1), \\ 0 & \text{otherwise,} \end{cases} \quad \text{and } \nabla \cdot h + \partial_3 m_3 = 0 \text{ in } \mathbb{R}^3 \tag{17}$$

for use in the following approximation lemma. As discussed for the other energies, we can also split the minimization in $h$ and $m_3$, replace the second term by the expression

$$\begin{aligned}
&\int_{(-l,l)^2\times\mathbb{R}} \left|(|\nabla'|^2+\varepsilon^2|\partial_3|^2)^{-1/2}\partial_3 m_3\right|^2 dx \\
&= \min\Big\{ \int_{(-l,l)^2\times\mathbb{R}} |\begin{pmatrix} h' \\ \frac{1}{\varepsilon}h_3\end{pmatrix}|^2 dx \,\Big|\, h:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}^3 \text{ is } (-l,l)^2\text{-periodic in } x', \\
&\qquad \nabla\cdot h + \partial_3 m_3 = 0 \text{ in } \mathbb{R}^3\Big\}
\end{aligned}$$

and write $e_{l,\varepsilon}(m_3)$ to emphasize that the minimal energy depends only on $m_3$.

**Lemma 3.** *Fix $m_3$ and the associated $h'$ in the admissible class in the minimization problem in for $e^p(l)$, i.e.*

$$m_3:\mathbb{R}^3\to\mathbb{R},\ h':\mathbb{R}^3\to\mathbb{R}^2, \quad (-l,l)^2\text{-periodic},$$
$$m_3^2 = \begin{cases} 1 & \text{if } x_3\in(-1,1), \\ 0 & \text{otherwise,}\end{cases} \qquad \nabla'\cdot h' + \partial_3 m_3 = 0.$$

*Then there exists a sequence $\{(m_3^{(\varepsilon)}, h^{(\varepsilon)})\}_{\varepsilon\downarrow 0}$ satisfying* (17) *such that*

$$\limsup_{\varepsilon\ll 1} e_{\varepsilon,l}(m_3^{(\varepsilon)}, h^{(\varepsilon)}) \le e_l(m_3,h'). \tag{18}$$

We postpone the proof of this lemma and first present the path from this approximation to the desired result of Theorem 4, part 1.

*Proof of Theorem 4, part 1.* Fix an $\alpha\ll 1$. Using Proposition 4 (see Section 7) with the slight generalization of Remark 3 we obtain from $m_3^{(\varepsilon)}$ functions $m_3^{(\varepsilon,\delta)}$ such that

$$m_3^{(\varepsilon,\delta)} \text{ is } (-l,l)^2 \text{ periodic in } x' \text{ and } (m_3^{(\varepsilon,\delta)})^2 \begin{cases} \le 1 & \text{for } x_3\in(-1,1), \\ =0 & \text{otherwise}\end{cases}$$

such that

$$\begin{aligned}
\delta\int_{(-l,l)^2\times(-1,1)} \frac{1}{1-(m_3^{(\varepsilon,\delta)})^2}\left|\begin{pmatrix}\nabla' \\ \varepsilon\partial_3\end{pmatrix} m_3^{(\varepsilon,\delta)}\right|^2 dx + \frac{1}{\delta}\int_{(-l,l)^2\times(-1,1)} 1-(m_3^{(\varepsilon,\delta)})^2\, dx \\
\le (1+\alpha)2\int_{[-l,l)^2\times(-1,1)}\left|\begin{pmatrix}\nabla' \\ \varepsilon\partial_3\end{pmatrix} m_3^{(\varepsilon)}\right| dx
\end{aligned} \tag{19}$$

and

$$\int_{(-l,l)^2\times(-1,1)} (m_3^{(\varepsilon,\delta)} - m_3^{(\varepsilon)})^2 dx \le C(\alpha)\delta\int_{[-l,l)^2\times(-1,1)}\left|\begin{pmatrix}\nabla' \\ \varepsilon\partial_3\end{pmatrix} m_3^{(\varepsilon)}\right| dx. \tag{20}$$

We then set

$$m_1^{(\varepsilon,\delta)} = \begin{cases} \sqrt{1-(m_3^{(\varepsilon,\delta)})^2} & \text{for } x_3\in(-1,1), \\ 0 & \text{otherwise,}\end{cases}$$

and

$$m_2^{(\varepsilon,\delta)} \equiv 0$$

so that

$$\left|m^{(\varepsilon,\delta)}\right|^2 = \begin{cases} 1 & \text{for } x_3 \in (-1,1), \\ 0 & \text{otherwise.} \end{cases}$$

We also set

$$h^{(\varepsilon,\delta)\prime} = h^{(\varepsilon)\prime} - \frac{1}{\varepsilon} m^{(\varepsilon,\delta)\prime} \qquad \text{and} \qquad h_3^{(\varepsilon,\delta)} = h_3^{(\varepsilon)} + m_3^{(\varepsilon)} - m_3^{(\varepsilon,\delta)},$$

so that (17) turns into

$$\nabla' \cdot (h^{(\varepsilon,\delta)\prime} + \frac{1}{\varepsilon} m^{(\varepsilon,\delta)\prime}) + \partial_3 (h_3^{(\varepsilon,\delta)} + m_3^{(\varepsilon,\delta)}) = 0$$

in $\mathbb{R}^3$ and see that $(m^{(\varepsilon,\delta)}, h^{(\varepsilon,\delta)})$ is admissible for $e^p(\delta,\varepsilon,l)$. We rewrite the first two terms of the energy using $|m|^2 = 1$ and $m_2 = 0$

$$\begin{aligned}
&\delta \int_{(-l,l)^2\times(-1,1)} \left| \left(\begin{smallmatrix} \nabla' \\ \varepsilon\partial_3 \end{smallmatrix}\right) m^{(\varepsilon,\delta)} \right|^2 dx + \frac{1}{\delta} \int_{(-l,l)^2\times(-1,1)} |m^{(\varepsilon,\delta)\prime}|^2 dx + \int_{(-l,l)^2\times\mathbb{R}} \left| \left(\begin{smallmatrix} h^{(\varepsilon,\delta)\prime} \\ \frac{1}{\varepsilon} h_3^{(\varepsilon,\delta)} \end{smallmatrix}\right) \right|^2 dx \\
&= \delta \int_{(-l,l)^2\times(-1,1)} \frac{1}{1-(m_3^{(\varepsilon,\delta)})^2} \left| \left(\begin{smallmatrix} \nabla' \\ \varepsilon\partial_3 \end{smallmatrix}\right) m_3^{(\varepsilon,\delta)} \right|^2 dx + \frac{1}{\delta} \int_{(-l,l)^2\times(-1,1)} 1 - (m_3^{(\varepsilon,\delta)})^2 \, dx \\
&\quad + \int_{(-l,l)^2\times\mathbb{R}} \left| \left(\begin{smallmatrix} h^{(\varepsilon,\delta)\prime} \\ \frac{1}{\varepsilon} h_3^{(\varepsilon,\delta)} \end{smallmatrix}\right) \right|^2 dx
\end{aligned}$$

and apply Young's inequality in the form $(a+b)^2 \le (1+\alpha)a^2 + (1+\alpha^{-1})b^2$ to split the field term and obtain

$$\begin{aligned}
&\le \delta \int_{(-l,l)^2\times(-1,1)} \frac{1}{1-(m_3^{(\varepsilon,\delta)})^2} \left| \left(\begin{smallmatrix} \nabla' \\ \varepsilon\partial_3 \end{smallmatrix}\right) m_3^{(\varepsilon,\delta)} \right|^2 dx + \frac{1}{\delta} \int_{(-l,l)^2\times(-1,1)} 1 - (m_3^{(\varepsilon,\delta)})^2 \, dx \\
&\quad + (1+\alpha) \int_{(-l,l)^2\times\mathbb{R}} \left| \left(\begin{smallmatrix} h^{(\varepsilon)\prime} \\ \frac{1}{\varepsilon} h_3^{(\varepsilon)} \end{smallmatrix}\right) \right|^2 dx + \frac{C}{\alpha} \int_{(-l,l)^2\times(-1,1)} \left| \left(\begin{smallmatrix} \frac{1}{\varepsilon} m^{(\varepsilon,\delta)\prime} \\ \frac{1}{\varepsilon}(m_3^{(\varepsilon)} - m_3^{(\varepsilon,\delta)}) \end{smallmatrix}\right) \right|^2 dx,
\end{aligned}$$

expanding the last integrand and applying (19) and (20) we estimate

$$\begin{aligned}
&= \delta \int_{(-l,l)^2\times(-1,1)} \frac{1}{1-(m_3^{(\varepsilon,\delta)})^2} \left| \left(\begin{smallmatrix} \nabla' \\ \varepsilon\partial_3 \end{smallmatrix}\right) m_3^{(\varepsilon,\delta)} \right|^2 dx + \frac{1}{\delta} \int_{(-l,l)^2\times(-1,1)} 1 - (m_3^{(\varepsilon,\delta)})^2 \, dx \\
&\quad + (1+\alpha) \int_{(-l,l)^2\times\mathbb{R}} \left| \left(\begin{smallmatrix} h^{(\varepsilon)\prime} \\ \frac{1}{\varepsilon} h_3^{(\varepsilon)} \end{smallmatrix}\right) \right|^2 dx \\
&\quad + \frac{C(\alpha)}{\varepsilon^2} \left( \int_{(-l,l)^2\times(-1,1)} 1 - (m_3^{(\varepsilon,\delta)})^2 dx + \int_{(-l,l)^2\times(-1,1)} (m_3^{(\varepsilon,\delta)} - m_3^{(\varepsilon)})^2 dx \right) \\
&\overset{(19),(20)}{\le} \left(1 + \alpha + C(\alpha)\frac{\delta}{\varepsilon^2}\right) \left( 2 \int_{[-l,l)^2\times(-1,1)} \left| \left(\begin{smallmatrix} \nabla' \\ \varepsilon\partial_3 \end{smallmatrix}\right) m_3^{(\varepsilon)} \right| dx + \int_{(-l,l)^2\times\mathbb{R}} \left| \left(\begin{smallmatrix} h^{(\varepsilon)\prime} \\ \frac{1}{\varepsilon} h_3^{(\varepsilon)} \end{smallmatrix}\right) \right|^2 dx \right) \\
&= \left(1 + \alpha + C(\alpha)\frac{\delta}{\varepsilon^2}\right) 4l^2 e_{l,\varepsilon}(m_3^{(\varepsilon)}, h^{(\varepsilon)}).
\end{aligned}$$

Combining this estimate in the limit $\delta \ll \varepsilon^2 \ll 1$ with (18) from Lemma 3 we conclude

$$\limsup_{\delta\ll\varepsilon^2\ll1} e^p(\delta,\varepsilon,l) \le \limsup_{\varepsilon^2\ll1} (1+\alpha)\, e_{l,\varepsilon}(m_3^{(\varepsilon)}, h^{(\varepsilon)}) \overset{(18)}{\le} (1+\alpha) e_l(m_3, h).$$

As $0 < \alpha \ll 1$ was arbitrary, we obtain the desired conclusion by using a diagonal sequence for the above limit relation and $\alpha \downarrow 0$. □

*Proof of Lemma 3.* Before we begin with the proof in full technical detail, let us point out the key ideas. In order to approximate $m_3$ with functions having some regularity in $x_3$-direction, the first thing that comes to mind is taking a convolution. This, however, does not play well with the requirement that the third component of the magnetization has unit length because the other two components vanish. So instead we approximate $m_3$ by piecewise (in $x_3$-direction) constant functions and obtain some $BV$-regularity also in this direction in the following way: For a third component in the $BV$-norm, we need to control the $L^1$-norm (with respect to the Hausdorff measure) of the jumps. The field term only yields a bound in the $H^{-1}$-norm, so we have to resort to interpolation (using Lemma 4) with the $x'$-perimeter over which we have control. The jump norm is small in $L^2$ (and thus also $L^1$) if the weight $\varepsilon$ is small compared to the discretization lengthscale $\tau$.

We incur, however, the problem that now the field energy measured as $\int ||\nabla'|^{-1}\partial_3 m_3|^2 dx$ is infinite in the presence of jumps. We thus need to allow an $\varepsilon$-small third component in the field term, i.e. introduce a $\partial_3$-term in the inverted operator (cf. (10)). As the jumps are essentially a surface phenomenon, we want, roughly speaking, to control the $H^{-1/2}$-norm. To that end we need to interpolate again between the $H^{-1}$-norm on a slowly changing component and the domain perimeter or, more precisely, the $L^2$-norm on the oscillations of short wave length (this happens on the level of Fourier series in (28)).

We begin with a few preparations in order to be able to define $m_3^{(\varepsilon)}$. Let us denote by $\mathcal{P}_\lambda$ the projection on the Fourier modes $n'$ with $\pi\lambda|\frac{n'}{l}| \geq 1$, i.e.

$$\mathcal{F}'(\mathcal{P}_\lambda\zeta)(n') = \begin{cases} (\mathcal{F}'\zeta)(n') & \text{if } \pi\lambda|\frac{n'}{l}| \geq 1, \\ 0 & \text{otherwise}, \end{cases} \tag{21}$$

where $\mathcal{F}'(\zeta)(n')$ is the Fourier coefficient

$$\mathcal{F}'(\zeta)(n') = \frac{1}{2l}\int_{(-l,l)^2} \exp(-\pi i n' \cdot \frac{x'}{l})\zeta(x')dx'.$$

Hence $\mathcal{P}_\lambda\zeta$ only sees the (horizontal) wavelengths smaller than $\lambda$.

The Fourier space representation of negative Sobolev norm appearing in the sharp interface field energy is

$$\int_{(-l,l)^2\times\mathbb{R}} |h'|^2 dx = \int_{(-l,l)^2\times\mathbb{R}} \left||\nabla'|^{-1}\partial_3 m_3\right|^2 dx = \int_{\mathbb{R}} \sum_{n'\in\mathbb{Z}^2} \frac{l^2}{\pi^2|n'|^2}|(\mathcal{F}'\partial_3 m_3)(n')|^2 dx_3,$$

see also the appendix. We need to make more precise the notion that small lengthscale oscillations in the magnetization, for our purposes $\mathcal{P}_\lambda\partial_3 m_3$ with $\lambda$ small, contribute little to the field energy. Note that an admissible $m_3$ cannot be constant in $x_3$-direction. As such we have

$$\int_{(-l,l)^2\times\mathbb{R}} \left||\nabla'|^{-1}\partial_3 m_3\right|^2 dx > 0,$$

and thus

$$\int_{(-l,l)^2\times\mathbb{R}} \left||\nabla'|^{-1}\mathcal{P}_\lambda\partial_3 m_3\right|^2 dx \leq (\omega(\lambda))^2 \int_{(-l,l)^2\times\mathbb{R}} \left||\nabla'|^{-1}\partial_3 m_3\right|^2 dx \tag{22}$$

with some modulus function $\omega > 0$ satisfying $\lim_{\lambda\downarrow 0}\omega(\lambda) = 0$ and depending only on $m_3$. We rewrite (22) as

$$\frac{1}{\omega(\lambda)}\int_{(-l,l)^2\times\mathbb{R}} \left||\nabla'|^{-1}\mathcal{P}_\lambda\partial_3 m_3\right|^2 dx \leq \omega(\lambda)4l^2 e_l(m_3).$$

We wish to find good layers to introduce the discontinuities in our envisioned $x_3$-piecewise constant approximation. This means that we want to limit the "slice" energy in these layers. Still integrating over all the domain we note that replacing $x_3$-derivatives by difference quotients does not enlarge the norms involved, thus

$$\int_{-\infty}^{+\infty}\left(2\int_{(-l,l)^2}|\nabla' m_3|\,dx' + \int_{(-l,l)^2}\left||\nabla'|^{-1}\frac{1}{\tau}(m_3(x',x_3+\tau)-m_3(x',x_3))\right|^2 dx'\right.$$
$$\left.+\frac{1}{\omega(\lambda)}\int_{(-l,l)^2}\left||\nabla'|^{-1}\frac{1}{\tau}\mathcal{P}_\lambda(m_3(x',x_3+\tau)-m_3(x',x_3))\right|^2 dx'\right)dx_3$$
$$\le (1+\omega(\lambda))4l^2 e_l(m_3,h').$$

We are now able to select a set of slices that is good for the energy on the left hand side of this inequality. For any fixed $N\in\mathbb{N}$ we set $\tau=\frac{1}{N}$. By Fubini's theorem not all slices can be above average and so there exists a $x_3^0\in(0,\tau)$ depending only on $m_3$, $\lambda$ and $N$ such that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\tau\left(2\int_{(-l,l)^2}\left|\nabla' m_3^{(k)}\right|dx' + \int_{(-l,l)^2}\left||\nabla'|^{-1}\frac{1}{\tau}[m_3]^{(k)}\right|^2 dx'\right.\\
&\qquad\left.+\frac{1}{\omega(\lambda)}\int_{(-l,l)^2}\left||\nabla'|^{-1}\frac{1}{\tau}\mathcal{P}_\lambda[m_3]^{(k)}\right|^2 dx'\right)\\
\le\;& \sum_{k\in\mathbb{Z}}\tau\fint_{(0,\tau)}\left(2\int_{(-l,l)^2}\left|\nabla' m_3^{(k)}\right|dx' + \int_{(-l,l)^2}\left||\nabla'|^{-1}\frac{1}{\tau}[m_3]^{(k)}\right|^2 dx'\right.\\
&\qquad\left.+\frac{1}{\omega(\lambda)}\int_{(-l,l)^2}\left||\nabla'|^{-1}\frac{1}{\tau}\mathcal{P}_\lambda[m_3]^{(k)}\right|^2 dx'\right)dx_3\\
\le\;& (1+\omega(\lambda))4l^2 e_l(m_3).
\end{aligned}\tag{23}$$

Here we use the abbreviations $m_3^{(k)}(x')=m_3(x',k\tau+x_3^0)$ and $[m_3]^{(k)}=m_3^{(k+1)}-m_3^{(k)}$. To finally obtain a candidate for $m_3^{(\varepsilon)}$ we take the piecewise constant (w.r.t. $x_3$) interpolant

$$\tilde{m}_3(x',x_3)=m_3^{(k)}(x')\qquad\text{for } x_3\in[k\tau,(k+1)\tau).$$

Note that $\tilde{m}_3$ is admissible for the $\varepsilon$-energy $e_{\varepsilon,l}$. We want to estimate $e_{\varepsilon,l}(\tilde{m}_3,\tilde{h})$ with $\tilde{h}$ minimal for given $m_3$. More precisely we are going to use an equivalent representation of the field energy. To prepare well, we use the interpolation estimate of Lemma 4 for $(-l,l)^2$-periodic $\varphi$

$$\left(\int_{(-l,l)^2}|\mathcal{P}_\lambda\varphi|^2dx'\right)^{1/2}\le C\left(\int_{[-l,l)^2}|\nabla'\varphi|dx'\right)^{1/3}\left(\sup_{(-l,l)^2}|\varphi|\right)^{1/3}\left(\int_{(-l,l)^2}\left||\nabla'|^{-1}\mathcal{P}_\lambda\varphi\right|^2dx'\right)^{1/6}$$

and conclude that for $\lambda\ll 1$ and small enough such that $\omega(\lambda)\le C$

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\int_{(-l,l)^2}|\mathcal{P}_\lambda[m_3]^{(k)}|^2dx'\\
\le\;& C\sum_{k\in\mathbb{Z}}\left(\int_{[-l,l)^2}|\nabla' m_3^{(k)}|dx'+\int_{[-l,l)^2}|\nabla' m_3^{(k+1)}|dx'\right)^{2/3}\left(\int_{(-l,l)^2}\left||\nabla'|^{-1}\mathcal{P}_\lambda[m_3]^{(k)}\right|^2dx'\right)^{1/3}\\
\le\;& C\left(\sum_{k\in\mathbb{Z}}\int_{[-l,l)^2}|\nabla' m_3^{(k)}|dx'\right)^{2/3}\left(\sum_{k\in\mathbb{Z}}\int_{(-l,l)^2}\left||\nabla'|^{-1}\mathcal{P}_\lambda[m_3]^{(k)}\right|^2dx'\right)^{1/3}\\
\overset{(23)}{\le}\;& C\left(\frac{1}{\tau}4l^2e_l(m_3)\right)^{2/3}\left(\omega(\lambda)\tau 4l^2e_l(m_3)\right)^{1/3}=C\omega(\lambda)^{1/3}\frac{1}{\tau^{1/3}}4l^2e_l(m_3).
\end{aligned}\tag{24}$$

Taking into account that $|[m_3]^{(k)}|$ is either 0 or at least 1 (in fact 0 or 2 except at the boundary), we likewise have

$$\sum_{k\in\mathbb{Z}} \int_{[-l,l)^2} |[m_3]^{(k)}|\, dx' \le \sum_{k\in\mathbb{Z}} \int_{(-l,l)^2} |[m_3]^{(k)}|^2 dx' \le C\frac{1}{\tau^{1/3}} 4l^2 e_l(m_3). \tag{25}$$

With these preparations we turn to estimate $e_l(\tilde{m}_3)$. We first bound the surface term

$$\begin{aligned} 2\int_{[-l,l)^2\times\mathbb{R}} \left| \left(\begin{smallmatrix} \nabla' \\ \varepsilon\partial_3 \end{smallmatrix}\right) \tilde{m}_3 \right| dx &\le 2\int_{[-l,l)^2\times\mathbb{R}} |\nabla' \tilde{m}_3| dx + 2\varepsilon \int_{[-l,l)^2\times\mathbb{R}} |\partial_3 \tilde{m}_3|\, dx \\ &= 2\sum_{k\in\mathbb{Z}} \tau \int_{[-l,l)^2} |\nabla' m_3^{(k)}| dx' + 2\varepsilon \sum_{k\in\mathbb{Z}} \int_{(-l,l)^2} |[m_3]^{(k)}| dx' \\ &\overset{(25)}{\le} \sum_{k\in\mathbb{Z}} \tau 2 \int_{[-l,l)^2} |\nabla' m_3^{(k)}| dx' + C\frac{\varepsilon}{\tau^{1/3}} 4l^2 e_l(m_3). \end{aligned} \tag{26}$$

To tackle the field term we take the Fourier series in $x'$ and the Fourier transform in $x_3$. Writing $G_\alpha(z_3) = \frac{1}{\alpha} G\left(\frac{z_3}{\alpha}\right)$ and $G(\hat{z}_3) = \frac{1}{2}\exp(-|\hat{z}_3|)$, the field term is

$$\begin{aligned} &\int_{(-l,l)^2\times\mathbb{R}} \left| (|\nabla'|^2 + \varepsilon^2|\partial_3|^2)^{-1/2} \partial_3 \tilde{m}_3 \right|^2 dx \\ &= \int_{\mathbb{R}} \sum_{n'\in\mathbb{Z}^2} \frac{1}{\pi^2(|n'|^2/l^2 + 4\varepsilon^2\xi^2)} \left| \int_{\mathbb{R}} \exp(-2\pi i \xi x_3) \partial_3 \mathcal{F}'(\tilde{m}_3)(n', x_3) dx_3 \right|^2 d\xi \\ &= \int_{\mathbb{R}} \sum_{n'\in\mathbb{Z}^2} \frac{1}{\pi^2(|n'|^2/l^2 + 4\varepsilon^2\xi^2)} \int_{\mathbb{R}} \int_{\mathbb{R}} \exp(-2\pi i \xi x_3) \partial_3 \mathcal{F}'(\tilde{m}_3)(n', x_3) \\ &\qquad \overline{\exp(-2\pi i \xi y_3) \partial_3 \mathcal{F}'(\tilde{m}_3)(n', y_3)} dx_3 dy_3 d\xi \\ &= \sum_{n'\in\mathbb{Z}^2} \frac{l^2}{\pi^2|n'|^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \frac{1}{1 + 4\varepsilon^2 l^2 \xi^2/|n'|^2} \exp(-2\pi i \xi (x_3 - y_3)) d\xi \right) \\ &\qquad \partial_3 \mathcal{F}'(\tilde{m}_3)(n', x_3) \overline{\partial_3 \mathcal{F}'(\tilde{m}_3)(n', y_3)} dx_3 dy_3 \\ &= \sum_{n'\in\mathbb{Z}^2} \frac{l^2}{\pi^2|n'|^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left( \frac{\pi}{2\varepsilon l/|n'|} \exp(-2\pi |x_3 - y_3||n'|/2\varepsilon l) \right) \\ &\qquad \partial_3 \mathcal{F}'(\tilde{m}_3)(n', x_3) \overline{\partial_3 \mathcal{F}'(\tilde{m}_3)(n', y_3)} dx_3 dy_3 \\ &= \sum_{n'\in\mathbb{Z}^2} \frac{l^2}{\pi^2|n'|^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} G_{\frac{\varepsilon l}{\pi|n'|}}(x_3 - y_3) \partial_3(\mathcal{F}'(\tilde{m}_3))(n', x_3) \overline{\partial_3(\mathcal{F}'(\tilde{m}_3))(n', y_3)} dx_3 dy_3. \end{aligned}$$

The above calculation is valid for smooth $\tilde{m}_3$ and by approximation also for our piecewise constant

$\tilde m_3(x',\cdot)$. In this case the two integrals on the right hand side are in fact sums, thus

$$\begin{aligned}
&\int_{(-l,l)^2\times\mathbb{R}} \left|(|\nabla'|^2+\varepsilon^2|\partial_3|^2)^{-1/2}\partial_3\tilde m_3\right|^2 dx \\
=&\sum_{n'\in\mathbb{Z}^2}\frac{l^2}{\pi^2|n'|^2}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} G_{\frac{\varepsilon l}{\pi|n'|}}((j-k)\tau)\mathcal{F}'([m_3]^{(j)})(n')\overline{\mathcal{F}'([m_3]^{(k)})(n')} \\
=&\sum_{n'\in\mathbb{Z}^2}\frac{l^2}{\pi^2|n'|^2}\frac{1}{\tau}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} G_{\frac{\varepsilon l}{\pi\tau|n'|}}(j-k)\mathcal{F}'([m_3]^{(j)})(n')\overline{\mathcal{F}'([m_3]^{(k)})(n')} \\
\le&\sum_{n'\in\mathbb{Z}^2}\frac{l^2}{\pi^2|n'|^2}\frac{1}{\tau}\left(\sum_{j\in\mathbb{Z}} G_{\frac{\varepsilon l}{\pi\tau|n'|}}(j)\right)\left(\sum_{k\in\mathbb{Z}}|\mathcal{F}'([m_3]^{(k)})(n')|^2\right).
\end{aligned}$$

Using the inequality $\exp(1/\alpha)\ge 1+1/\alpha$ we observe that

$$\sum_{j\in\mathbb{Z}} G_\alpha(j)=\frac{1}{2\alpha}\left(-1+2\sum_{j=0}^{\infty}\exp\left(-\frac{1}{\alpha}\right)^j\right)=\frac{1}{2\alpha}\frac{1+\exp(-1/\alpha)}{1-\exp(-1/\alpha)}\le 1+\frac{1}{2\alpha},$$

and so

$$\begin{aligned}
&\int_{(-l,l)^2\times\mathbb{R}} \left|(|\nabla'|^2+\varepsilon^2|\partial_3|^2)^{-1/2}\partial_3\tilde m_3\right|^2 dx \\
\le&\ \frac{1}{\tau}\sum_{n'\in\mathbb{Z}^2}\frac{l^2}{\pi^2|n'|^2}\sum_{k\in\mathbb{Z}}|\mathcal{F}'([m_3]^{(k)})(n')|^2+\frac{1}{2\varepsilon}\sum_{n'\in\mathbb{Z}^2}\frac{l}{\pi|n'|}\sum_{k\in\mathbb{Z}}|\mathcal{F}'([m_3]^{(k)})(n')|^2. \qquad (27)
\end{aligned}$$

The second sum is an $H^{-1/2}$-norm which we estimate by interpolating between the $H^{-1}$-norm and the $L^2$-norm for high wave numbers. More precisely, we estimate splitting the second sum

$$\begin{aligned}
&\sum_{n'\in\mathbb{Z}^2}\frac{l}{\pi|n'|}\sum_{k\in\mathbb{Z}}|\mathcal{F}'([m_3]^{(k)})(n')|^2 \\
=&\sum_{\substack{n'\in\mathbb{Z}^2\\ \pi\lambda|n'|/l<1}}\frac{l}{\pi|n'|}\sum_{k\in\mathbb{Z}}|\mathcal{F}'([m_3]^{(k)})(n')|^2+\sum_{\substack{n'\in\mathbb{Z}^2\\ \pi\lambda|n'|/l\ge1}}\frac{l}{\pi|n'|}\sum_{k\in\mathbb{Z}}|\mathcal{F}'([m_3]^{(k)})(n')|^2 \\
\le&\ \frac{1}{\lambda}\sum_{n'\in\mathbb{Z}^2}\frac{l^2}{\pi^2|n'|^2}\sum_{k\in\mathbb{Z}}|\mathcal{F}'([m_3]^{(k)})(n')|^2+\lambda\sum_{\substack{n'\in\mathbb{Z}^2\\ \pi\lambda|n'|/l\ge1}}\sum_{k\in\mathbb{Z}}|\mathcal{F}'([m_3]^{(k)})(n')|^2. \qquad (28)
\end{aligned}$$

Thus using (23) and (24) we obtain for $\lambda$ small enough such that $\omega(\lambda)\le C$

$$\begin{aligned}
&\int_{(-l,l)^2\times\mathbb{R}} \left|(|\nabla'|^2+\varepsilon^2|\partial_3|^2)^{-1/2}\partial_3\tilde m_3\right|^2 dx \\
\overset{(27),(28)}{\le}&\left(\frac{1}{\tau}+\frac{1}{2\varepsilon\lambda}\right)\sum_{n'\in\mathbb{Z}^2}\frac{l^2}{\pi^2|n'|^2}\sum_{k\in\mathbb{Z}}|\mathcal{F}'([m_3]^{(k)})(n')|^2+\frac{\lambda}{2\varepsilon}\sum_{\substack{n'\in\mathbb{Z}^2\\ \lambda\pi|n'|/l\ge1}}\sum_{k\in\mathbb{Z}}|\mathcal{F}'([m_3]^{(k)})(n')|^2 \\
=&\left(1+\frac{\tau}{2\varepsilon\lambda}\right)\sum_{k\in\mathbb{Z}}\tau\int_{(-l,l)^2}\left|\frac{1}{\tau}|\nabla'|^{-1}[m_3]^{(k)}\right|^2 dx'+\frac{\lambda}{2\varepsilon}\sum_{k\in\mathbb{Z}}\int_{(-l,l)^2}\left|\mathcal{P}_\lambda[m_3]^{(k)}\right|^2 dx' \\
\overset{(23),(24)}{\le}&\sum_{k\in\mathbb{Z}}\tau\int_{(-l,l)^2}\left|\frac{1}{\tau}|\nabla'|^{-1}[m_3]^{(k)}\right|^2 dx'+C\frac{\tau}{\varepsilon\lambda}4l^2e_l(m_3)+C\frac{\lambda\omega(\lambda)^{1/3}}{\varepsilon\tau^{1/3}}4l^2e_l(m_3). \qquad (29)
\end{aligned}$$

Combining (26) with (29) and employing (23) we see that

$$
\begin{aligned}
2\int_{[-l,l)^2\times\mathbb{R}} \left|\left(\begin{smallmatrix}\nabla'\\ \varepsilon\partial_3\end{smallmatrix}\right)\tilde m_3\right| dx + \int_{(-l,l)^2\times\mathbb{R}} \left|(|\nabla'|^2+\varepsilon^2|\partial_3|^2)^{-1/2}\partial_3\tilde m_3\right|^2 dx \\
\overset{(26),(29)}{\le} \quad & \sum_{k\in\mathbb{Z}} \tau\left(2\int_{(-l,l)^2}\left|\nabla' m_3^{(k)}\right|^2 dx' + \int_{(-l,l)^2}\left|\frac{1}{\tau}|\nabla'|^{-1}[m_3]^{(k)}\right|^2 dx'\right) \\
& + C\left(\frac{\varepsilon}{\tau^{1/3}}+\frac{\tau}{\varepsilon\lambda}+\frac{\lambda\omega(\lambda)^{1/3}}{\varepsilon\tau^{1/3}}\right)4l^2 e_l(m_3) \\
\overset{(23)}{\le} \quad & 4l^2 e_l(m_3) + C\left(\omega(\lambda)+\frac{\varepsilon}{\tau^{1/3}}+\frac{\tau}{\varepsilon\lambda}+\frac{\lambda\omega(\lambda)^{1/3}}{\varepsilon\tau^{1/3}}\right)4l^2 e_l(m_3). \qquad (30)
\end{aligned}
$$

In the last inequality we have used our choice of "good slices" again. To finish the proof, we need to arrange for the second term to vanish in the limit, so choosing $\tau = M^3\varepsilon^3$ and $\lambda = M^4\varepsilon^2$ we compute

$$
\omega(\lambda)+\frac{\varepsilon}{\tau^{1/3}}+\frac{\tau}{\varepsilon\lambda}+\frac{\lambda\omega(\lambda)^{1/3}}{\varepsilon\tau^{1/3}} = \omega(M^4\varepsilon^2)+\frac{1}{M}+\frac{1}{M}+M^3\omega(M^4\varepsilon^2)^{1/3}.
$$

Since

$$
\lim_{M\uparrow\infty}\lim_{\varepsilon\downarrow 0}\left(\omega(M^4\varepsilon^2)+\frac{1}{M}+M^3\omega(M^4\varepsilon^2)^{1/3}\right)=0
$$

we can select sequences $\{N^{(\varepsilon)}=\frac{1}{\tau^{(\varepsilon)}}\}_{\varepsilon\downarrow 0}$ and $\{\lambda^{(\varepsilon)}\}_{\varepsilon\downarrow 0}$ such that

$$
\lim_{\varepsilon\downarrow 0}\left(\omega(\lambda^{(\varepsilon)})+\frac{\varepsilon^{(\varepsilon)}}{(\tau^{(\varepsilon)})^{1/3}}+\frac{\tau^{(\varepsilon)}}{\varepsilon\lambda^{(\varepsilon)}}+\frac{\lambda^{(\varepsilon)}\omega(\lambda^{(\varepsilon)})^{1/3}}{\varepsilon(\tau^{(\varepsilon)})^{1/3}}\right)=0.
$$

Application of (30) for the corresponding sequence of $\tilde m_3^{(\varepsilon)}$ yields the assertion of the lemma. □

We now provide the interpolation inequality used in the proof of Lemma 3. It originally appeared in [CKO99, Lemma 2.3], but we wish to present a simplified argument here.

**Lemma 4.** *There exists a universal constant $C$ such that*

$$
\left(\int_{(-l,l)^2}|P\zeta|^2dx'\right)^{1/2} \le C\left(\sup_{(-l,l)^2}|\zeta|\right)^{1/3}\left(\int_{[-l,l)^2}|\nabla'\zeta|dx'\right)^{1/3}\left(\int_{(-l,l)^2}\left||\nabla'|^{-1}P\zeta\right|^2dx'\right)^{1/6}
$$

*for all $(-l,l)^2$-periodic $\zeta:\mathbb{R}^2\to\mathbb{R}$ and $P$ either be the identity or the projection $\mathcal{P}_\lambda$ on the Fourier modes as defined in* (21).

*Proof.* We fix a smooth convolution kernel $\varphi:\mathbb{R}^2\to\mathbb{R}$ with

$$
\varphi\ge 0,\quad \varphi=0 \text{ for } |x'|\ge 1,\quad \varphi(-x')=\varphi(x'),\text{ and}\quad \int_{\mathbb{R}^2}\varphi dx'=1
$$

and denote by subscript $\alpha$ the convolution with $\frac{1}{\alpha^2}\varphi(\frac{\cdot}{\alpha})$. Note that $P$ commutes with convolution and is indeed a projection in $L^2$.

We observe that for any $(-l,l)^2$-periodic $\zeta:\mathbb{R}^2\to\mathbb{R}$ we have

$$
\begin{aligned}
\int_{(-l,l)^2}|\zeta(x'+h')-\zeta(x')|^2dx' &\le \sup_{x'\in(-l,l)^2}|\zeta(x'+h')-\zeta(x')|\int_{(-l,l)^2}|\zeta(x'+h')-\zeta(x')|dx' \\
&\le 2\left(\sup_{(-l,l)^2}|\zeta|\right)\cdot|h'|\int_{[-l,l)^2}|\nabla'\zeta|dx'. \qquad (31)
\end{aligned}
$$

A standard convolution argument using Jensen's inequality shows

$$\begin{aligned}
\int_{(-l,l)^2} |\zeta-\zeta_\alpha|^2 dx' &\le \int_{\mathbb{R}^2} \frac{1}{\alpha^2}\varphi\left(\frac{y'}{\alpha}\right) \int_{(-l,l)^2} |\zeta(x')-\zeta(x'-y')|^2 dx' dy' \\
&\le \int_{\mathbb{R}^2} \frac{1}{\alpha^2}\varphi\left(\frac{y'}{\alpha}\right) dy' \alpha \sup_{h'} \left(\frac{1}{|h'|}\int_{(-l,l)^2} |\zeta(x')-\zeta(x'+h')|^2 dx'\right) \\
&= \alpha \sup_{h'} \left(\frac{1}{|h'|}\int_{(-l,l)^2} |\zeta(x')-\zeta(x'+h')|^2 dx'\right) \\
&\overset{(31)}{\le} 2\alpha \left(\sup_{(-l,l)^2} |\zeta|\right) \int_{[-l,l)^2} |\nabla'\zeta| dx'. \qquad (32)
\end{aligned}$$

By duality (see (106) in the appendix) the standard estimate

$$\begin{aligned}
\int_{(-l,l)^2} |\nabla'\psi_\alpha|^2 dx' &\le \int_{\mathbb{R}^2} \frac{1}{\alpha^3}\left|\nabla'\varphi\left(\frac{x'}{\alpha}\right)\right| dx' \int_{(-l,l)^2} |\psi|^2\, dx' \\
&\le C\frac{1}{\alpha^2}\int_{(-l,l)^2} |\psi|^2 dx', \qquad (33)
\end{aligned}$$

valid for all $(-l,l)^2$-periodic $\psi:\mathbb{R}^2\to\mathbb{R}$, entails

$$\begin{aligned}
\int_{(-l,l)^2} |(P\zeta)_\alpha|^2 dx' &= \int_{(-l,l)^2} (P\zeta)_{\alpha\alpha} P\zeta dx' \\
&\overset{(106)}{\le} \left(\int_{(-l,l)^2} |\nabla'(P\zeta)_{\alpha\alpha}|^2 dx'\right)^{1/2} \left(\int_{(-l,l)^2} \left||\nabla'|^{-1}(P\zeta)\right|^2 dx'\right)^{1/2} \\
&\overset{(33)}{\le} C\frac{1}{\alpha}\left(\int_{(-l,l)^2} |(P\zeta)_\alpha|^2 dx'\right)^{1/2} \left(\int_{(-l,l)^2} \left||\nabla'|^{-1}(P\zeta)\right|^2 dx'\right)^{1/2}
\end{aligned}$$

and thus

$$\int_{(-l,l)^2} |(P\zeta)_\alpha|^2 dx' \le C\frac{1}{\alpha^2}\int_{(-l,l)^2} \left||\nabla'|^{-1}(P\zeta)\right|^2 dx'. \qquad (34)$$

Note that convolution and the projection $P$ on Fourier modes are (pointwise) multiplications in Fourier space and thus commute, in particular $P\zeta-(P\zeta)_\alpha = P\zeta - P(\zeta_\alpha) = P(\zeta-\zeta_\alpha)$. Combining (32) and (34) after using the triangle inequality and the projection property of $P$ we estimate

$$\begin{aligned}
&\left(\int_{(-l,l)^2} |P\zeta|^2 dx'\right)^{1/2} \\
&\le \left(\int_{(-l,l)^2} |(P\zeta)_\alpha|^2 dx'\right)^{1/2} + \left(\int_{(-l,l)^2} |P\zeta-(P\zeta)_\alpha|^2 dx'\right)^{1/2} \\
&\le \left(\int_{(-l,l)^2} |(P\zeta)_\alpha|^2 dx'\right)^{1/2} + \left(\int_{(-l,l)^2} |\zeta-\zeta_\alpha|^2 dx'\right)^{1/2} \\
&\overset{(32),(34)}{\le} C\left(\frac{1}{\alpha}\left(\int_{(-l,l)^2} \left||\nabla'|^{-1}P\zeta\right|^2 dx'\right)^{1/2} + \alpha^{1/2}\left(\sup_{(-l,l)^2}|\zeta| \int_{[-l,l)^2} |\nabla'\zeta| dx'\right)^{1/2}\right).
\end{aligned}$$

We obtain the assertion of the lemma by choosing the optimal

$$\alpha = \left( \frac{\int_{(-l,l)^2} \left| |\nabla'|^{-1} P\zeta \right|^2 dx'}{\sup_{(-l,l)^2} |\zeta| \int_{[-l,l)^2} |\nabla'\zeta| dx'} \right)^{1/3}. \qquad \square$$

# 6 Asymptotic behavior of the energy

In this section we prove Theorem 2 by comparing the various energies introduced in Section 4. Most of the results needed in addition to Theorem 4 follow from the energy scaling and some elementary reflection and extension arguments. For the lower bound, we also need the interpolation inequality of Lemma 4, the upper bound is provided by an explicit construction. As we shall need to fix a few constants for later reference, we recall that $C$ denotes an arbitrary constant (universal or depending only on the parameters indicated in parantheses) that can change between any two occurrences, while the numbered constants $C_1$, $C_2$, etc. are fixed within this section.

**Lemma 5.** *The sharp interface energy per cross-section area on configurations with periodic boundary conditions is bounded from below, i.e.*

$$\liminf_{1 \ll l} e^p(l) > 0.$$

*Proof.* At the core of the proof is the interpolation estimate of Lemma 4. Fix an arbitrary $l$ and let $(m_3, h')$ be admissible for $e^p(l)$. Recall that

$$4l^2 e_l^p(m_3, h') = 2 \int_{[-l,l) \times (-1,1)} |\nabla' m_3| dx + \int_{(-l,l)^2 \times \mathbb{R}} |h'|^2 dx. \tag{35}$$

We estimate the second term from below by

$$\begin{aligned}
\int_{(-l,l)^2 \times \mathbb{R}} |h'|^2 dx &= \int_{-\infty}^{+\infty} \int_{(-l,l)^2} |h'|^2 dx' dx_3 \\
&\geq \int_{-\infty}^{+\infty} \int_{(-l,l)^2} \left| |\nabla'|^{-1} \partial_3 m_3 \right|^2 dx' dx_3 \\
&\geq \left(\frac{\pi}{2}\right)^2 \int_{-1}^{1} \int_{(-l,l)^2} \left| |\nabla'|^{-1} m_3 \right|^2 dx' dx_3,
\end{aligned}$$

where we use the Poincaré estimate on $(-1,1)$. For the first term in (35) we observe

$$\begin{aligned}
2 \int_{[-l,l)^2 \times (-1,1)} |\nabla' m_3| dx &= 2 \int_{-1}^{1} \int_{[-l,l)^2} |\nabla' m_3| dx' dx_3 \\
&\geq 2 \int_{-1}^{1} \sup_{(-l,l)^2} |m_3| \int_{[-l,l)^2} |\nabla' m_3| dx' dx_3,
\end{aligned}$$

because $|m_3| \leq 1$, and hence using Young's inequality and Lemma 4 we can estimate the energy from below as

$$\begin{aligned}4l^2e^p_l(m_3,h') &\geq \frac{1}{C}\int_{-1}^{1}\left(\sup_{(-l,l)^2}|m_3|\int_{[-l,l)^2}|\nabla' m_3|dx' + \int_{(-l,l)^2}\left||\nabla'|^{-1}m_3\right|^2dx'\right)dx_3\\
&\geq \frac{1}{C}\int_{-1}^{1}\left(\sup_{(-l,l)^2}|m_3|\int_{[-l,l)^2}|\nabla' m_3|dx'\right)^{2/3}\left(\int_{(-l,l)^2}\left||\nabla'|^{-1}m_3\right|^2dx'\right)^{1/3}dx_3\\
&\geq \frac{1}{C}\int_{-1}^{1}\int_{(-l,l)^2}m_3^2dx'dx_3 = \frac{1}{C}4l^2,\end{aligned}$$

that is

$$e^p_l(m_3,h') \geq \frac{1}{C}.$$

Since $(m_3,h')$ was an arbitrary admissible pair, $e^p(l) \geq \frac{1}{C}$, as claimed. □

**Lemma 6.** *In the setting of free boundary conditions, decoupling the passage to the limit $l \to \infty$ from the limits in $\delta$ and $\varepsilon$ does not increase the limiting energy of the minimizers. More precisely*

$$\liminf_{\delta\ll\varepsilon^2\ll 1\ll l} e^f(\varepsilon,\delta,l) \geq \limsup_{1\ll l}\liminf_{\delta\ll\varepsilon^2\ll 1} e^f(\varepsilon,\delta,l).$$

*Proof.* A key ingredient to establish the claim is

$$e^f(\varepsilon,\delta,Nl_0) \geq e^f(\varepsilon,\delta,l_0) \quad \text{for } N\in\mathbb{N}. \tag{36}$$

To establish this inequality, consider a minimizer $(m,h)$ for $e^f(\varepsilon,\delta,Nl_0)$ and decompose the domain $(-Nl_0,Nl_0)^2$ into $N^2$ squares $\{Q_n\}_{=\{0..N-1\}^2}$ of edge length $2l_0$. Denote by $(m_n,h_n)$ the restriction of $(m,h)$ onto $Q_n\times\mathbb{R}$ translated back to $(-l_0,l_0)^2\times\mathbb{R}$. As these are admissible in the minimization problem for $e^f(\varepsilon,\delta,l_0)$ and as the energy functional is translation invariant we conclude with

$$e_{\varepsilon,\delta,Nl_0}(m,h) = \sum_{n\in\{0,\ldots,N-1\}^2} e_{\varepsilon,\delta,Nl_0}(m_n,h_n)$$

that

$$(Nl_0)^2e^f(\varepsilon,\delta,Nl_0) \geq \sum_{n\in\{0,\ldots,N-1\}^2} l_0^2e^f(\varepsilon,\delta,l_0).$$

As a second item we need that

$$e^f(\varepsilon,\delta,l) \geq \left(\frac{\tilde{l}}{l}\right)^2 e^f(\varepsilon,\delta,\tilde{l}) \quad \text{for } l\geq\tilde{l}, \tag{37}$$

which is evident when considering that the restriction of a minimizer $(m,h)$ for $e^f(\varepsilon,\delta,l)$ to $(-\tilde{l},\tilde{l})$ is admissible for $e^f(\varepsilon,\delta,\tilde{l})$.

Now fix $l_0$ and let $l\geq l_0$ be arbitrary. Write $l = Nl_0 + r$ with $N\in\mathbb{N}$ and $r\in[0,l_0)$. By above estimates (36) and (37) we have

$$e^f(\varepsilon,\delta,l) \geq \left(\frac{Nl_0}{l}\right)^2 e^f(\varepsilon,\delta,Nl_0) \geq \left(\frac{l-l_0}{l}\right)^2 e^f(\varepsilon,\delta,l_0) = \left(1-\frac{l_0}{l}\right)^2 e^f(\varepsilon,\delta,l_0),$$

thus

$$\liminf_{\delta\ll\varepsilon^2\ll 1\ll l} e^f(\varepsilon,\delta,l) \geq \liminf_{\delta\ll\varepsilon^2\ll 1} e^f(\varepsilon,\delta,l_0),$$

and so as $l_0$ was arbitrary

$$\liminf_{\delta\ll\varepsilon^2\ll 1\ll l} e^f(\varepsilon,\delta,l) \geq \limsup_{1\ll l}\liminf_{\delta\ll\varepsilon^2\ll 1} e^f(\varepsilon,\delta,l). \qquad \square$$

**Lemma 7.** *In the setting of periodic boundary conditions, coupling the passage to the limit $l\to\infty$ with the limits in $\delta$ and $\varepsilon$ does not increase the limiting energy of the minimizers. More precisely*

$$\limsup_{\delta\ll\varepsilon^2\ll 1\ll l} e^p(\varepsilon,\delta,l) \leq \liminf_{1\ll l}\limsup_{\delta\ll\varepsilon^2\ll 1} e^p(\varepsilon,\delta,l).$$

*Proof.* Because $(-l,l)^2$-periodicity implies $(-Nl,Nl)^2$-periodicity and the resulting inclusion of the admissible classes we have

$$e^p(\varepsilon,\delta,Nl) \leq e^p(\varepsilon,\delta,l) \quad \text{for } N\in\mathbb{N}. \tag{38}$$

We claim that

$$e^p(\varepsilon,\delta,l) \leq \left(2\frac{l}{\tilde{l}}-1\right)^2 e^p(\varepsilon,\delta,\tilde{l}) \quad \text{for } l\geq\tilde{l}, \varepsilon^2\geq\delta. \tag{39}$$

Indeed, let $(\tilde{m},\tilde{h})$ be a minimizer for $e^p(\varepsilon,\delta,\tilde{l})$. We define an admissible $(m,h)$ for $e^p(\varepsilon,\delta,l)$ as follows: Let

$$\begin{aligned}
m(x',x_3) &= \tilde{m}\left(\frac{\tilde{l}}{l}x',x_3\right),\\
h'(x',x_3) &= \left(\frac{l}{\tilde{l}}\tilde{h}' + \left(\frac{l}{\tilde{l}}-1\right)\frac{1}{\varepsilon}\tilde{m}'\right)\left(\frac{\tilde{l}}{l}x',x_3\right), \text{ and}\\
h_3(x',x_3) &= \tilde{h}_3\left(\frac{\tilde{l}}{l}x',x_3\right).
\end{aligned}$$

Thus $h'$ is defined such that

$$\left(h'+\frac{1}{\varepsilon}m'\right)(x',x_3) = \frac{l}{\tilde{l}}\left(\tilde{h}'+\frac{1}{\varepsilon}\tilde{m}'\right)\left(\frac{\tilde{l}}{l}x',x_3\right),$$

ensuring

$$\left(\nabla'\cdot\left(h'+\frac{1}{\varepsilon}m'\right)\right)(x',x_3) = \left(\nabla'\cdot\left(\tilde{h}'+\frac{1}{\varepsilon}\tilde{m}'\right)\right)\left(\frac{\tilde{l}}{l}x',x_3\right),$$

i.e. the admissibility of $(m,h)$.

Furthermore using $\tilde{l}\leq l$ and $\delta\leq\varepsilon^2$ we have

$$\begin{aligned}
\frac{1}{4l^2}\delta\int_{(-l,l)^2\times(-1,1)}\left|\begin{pmatrix}\nabla'\\ \varepsilon\partial_3\end{pmatrix}m\right|^2 dx &= \frac{1}{4\tilde{l}^2}\delta\int_{(-\tilde{l},\tilde{l})^2\times(-1,1)}\left|\begin{pmatrix}\frac{\tilde{l}}{l}\nabla'\\ \varepsilon\partial_3\end{pmatrix}\tilde{m}\right|^2 dx\\
&\leq \frac{1}{4\tilde{l}^2}\delta\int_{(-\tilde{l},\tilde{l})^2\times(-1,1)}\left|\begin{pmatrix}\nabla'\\ \varepsilon\partial_3\end{pmatrix}\tilde{m}\right|^2 dx,
\end{aligned}$$

as well as

$$\frac{1}{4l^2}\frac{1}{\delta}\int_{(-l,l)^2\times(-1,1)}|m'|^2\,dx = \frac{1}{4\tilde{l}^2}\frac{1}{\delta}\int_{(-\tilde{l},\tilde{l})^2\times(-1,1)}|\tilde{m}'|^2\,dx,$$

and

$$
\begin{aligned}
&\frac{1}{4l^2}\int_{(-l,l)^2\times\mathbb{R}}\left|\begin{pmatrix} h' \\ \frac{1}{\varepsilon}h_3\end{pmatrix}\right|^2 dx \\
&= \frac{1}{4\tilde{l}^2}\int_{(-\tilde{l},\tilde{l})^2\times\mathbb{R}}\left|\begin{pmatrix} \frac{l}{\tilde{l}}\tilde{h}'+(\frac{l}{\tilde{l}}-1)\frac{1}{\varepsilon}\tilde{m}' \\ \frac{1}{\varepsilon}\tilde{h}_3\end{pmatrix}\right|^2 dx \\
&\le \left(\frac{l}{\tilde{l}}\right)^2 \frac{1}{4\tilde{l}^2}\int_{(-\tilde{l},\tilde{l})^2\times\mathbb{R}}\left|\begin{pmatrix} \tilde{h}' \\ \frac{1}{\varepsilon}\tilde{h}_3\end{pmatrix}\right|^2 dx \\
&\quad + 2\frac{l}{\tilde{l}}\left(\frac{l}{\tilde{l}}-1\right)\left(\frac{1}{4\tilde{l}^2}\int_{(-\tilde{l},\tilde{l})^2\times\mathbb{R}}|\tilde{h}'|^2dx\right)^{1/2}\left(\frac{1}{4\tilde{l}^2}\frac{1}{\varepsilon^2}\int_{(-\tilde{l},\tilde{l})^2\times(-1,1)}|\tilde{m}'|^2dx\right)^{1/2} \\
&\quad + \left(\frac{l}{\tilde{l}}-1\right)^2\frac{1}{4\tilde{l}^2}\frac{1}{\varepsilon^2}\int_{(-\tilde{l},\tilde{l})^2\times(-1,1)}|\tilde{m}'|^2dx \\
&\le \left(\frac{l}{\tilde{l}}\right)^2 \frac{1}{4\tilde{l}^2}\int_{(-\tilde{l},\tilde{l})^2\times\mathbb{R}}\left|\begin{pmatrix} \tilde{h}' \\ \frac{1}{\varepsilon}\tilde{h}_3\end{pmatrix}\right|^2 dx + \left(2\frac{l}{\tilde{l}}\left(\frac{l}{\tilde{l}}-1\right)+\left(\frac{l}{\tilde{l}}-1\right)^2\right)e^p(\varepsilon,\delta,\tilde{l}).
\end{aligned}
$$

Hence

$$
e^p(\varepsilon,\delta,l) \le \left(\left(\frac{l}{\tilde{l}}\right)^2 + 2\frac{l}{\tilde{l}}\left(\frac{l}{\tilde{l}}-1\right)+\left(\frac{l}{\tilde{l}}-1\right)^2\right)e^p(\varepsilon,\delta,\tilde{l}) = \left(2\frac{l}{\tilde{l}}-1\right)^2 e^p(\varepsilon,\delta,\tilde{l}),
$$

as claimed.

With (38) and (39) at our disposal, we are able to proceed as in Lemma 6. Fix $l_0$ and let $l \ge l_0$. We write

$$
l = Nl_0 + r \text{ with } N \in \mathbb{N} \text{ and } r \in [0, l_0).
$$

Then for $\varepsilon^2 \ge \delta$ we can estimate

$$
e^p(\varepsilon,\delta,l) \le \left(2\frac{l}{Nl_0}-1\right)^2 e^p(\varepsilon,\delta,Nl_0) \le \left(2\frac{l}{l-l_0}-1\right)^2 e^p(\varepsilon,\delta,l_0) = \left(\frac{l+l_0}{l-l_0}\right)^2 e^p(\varepsilon,\delta,l_0).
$$

Hence

$$
\limsup_{\delta\ll\varepsilon^2\ll 1\ll l} e^p(\varepsilon,\delta,l) \le \liminf_{\delta\ll\varepsilon^2\ll 1} e^p(\varepsilon,\delta,l_0),
$$

and again the assertion of the lemma follows because $l_0$ was arbitrary. □

**Lemma 8.** *The minimal sharp-interface energy per cross-section area amongst admissible configurations with free boundary conditions is bounded, i.e.*

$$
\limsup_{1\ll l} e^f(l) < \infty.
$$

*Proof.* We use four main estimates for the proof. First, completely analogous to (37) we have

$$
e^f(l) \le \left(\frac{\tilde{l}}{l}\right)^2 e^f(\tilde{l}) \quad \text{for } l \le \tilde{l}, \tag{40}
$$

then because the inclusion of the corresponding admissible classes we obviously have

$$
e^f(l) \le e^p(l). \tag{41}
$$

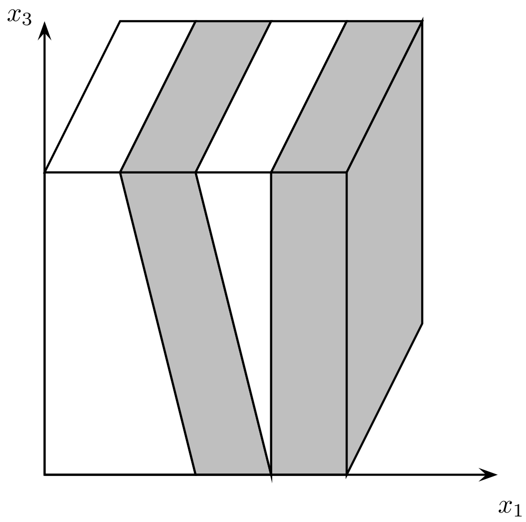


Figure 8: The refinement (constant in $x_2$)

Thirdly, as in (38) we have

$$e^p(N) \le e^p(1) \quad \text{for } N \in \mathbb{N}. \tag{42}$$

And as a fourth ingredient we need the estimate

$$e^p(1) < \infty. \tag{43}$$

To establish the latter, we have to construct $m_3 : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ such that

$$m_3 \text{ is } (-1,1)^2\text{-periodic in } x', \quad m_3^2 = \begin{cases} 1 & \text{if } x_3 \in (-1,1), \\ 0 & \text{otherwise,} \end{cases}$$

with

$$2\int_{[-1,1)^2\times(-1,1)} |\nabla' m_3| dx + \int_{(-1,1)^2\times\mathbb{R}} \big||\nabla'|^{-1}\partial_3 m_3\big|^2\, dx < \infty.$$

By symmetry and translation invariance, it suffices to construct $m_3 : \mathbb{R}^2 \times (0,1) \to \mathbb{R}$ such that

$$m_3 \text{ is } (-1,1)^2\text{-periodic in } x', \quad m_3^2 = 1, \text{ and } \quad m_3(\,\cdot\,,) \xrightarrow{x_3\uparrow 1} 0 \text{ (weakly) in } L^\infty((-1,1)^2)$$

with

$$2\int_{[-1,1)^2\times(0,1)} |\nabla' m_3| dx + \int_{(-1,1)^2\times(0,1)} \big||\nabla'|^{-1}\partial_3 m_3\big|^2\, dx < \infty.$$

Denote by $m_3^0 : \mathbb{R}^2 \to \mathbb{R}$ the $(-1,1)^2$-periodic function given by

$$m_3^0(x_1,x_2) = \operatorname{sign} x_1.$$

Obviously (see e.g. Figure 8), one can construct $m_3^{01} : \mathbb{R}^2 \times (0,1) \to \mathbb{R}$ such that

$$m_3^{01} \text{ is } (-1,1)\text{-periodic in } x', \quad (m_3^{01})^2 = 1,$$
$$m_3^{01}(x',0) = m_3^0(x'), \text{ and } \quad m_3^{01}(x',1) = m_3^0(2x'),$$

and

$$2\int_{[-1,1)^2\times(0,1)} |\nabla' m_3^{01}|dx + \int_{(-1,1)^2\times(0,1)} \left||\nabla'|^{-1}\partial_3 m_3^{01}\right|^2 dx < \infty.$$

We now glue rescaled versions of $m_3^{01}$. Let $\Delta_k = \theta^k(1-\theta)$ with $\theta \in (0,1)$ to be chosen later. Note that $\sum_{k=0}^{\infty} \Delta_k = 1$. Let $x_3^{(0)} = 0$ and $x_3^{(k+1)} = x_3^{(k)} + \Delta_k$ and define

$$m_3(x', x_3) = m_3^{01}\left(2^k x', \frac{x_3 - x_3^{(k)}}{\Delta_k}\right) \quad \text{for } x_3 \in (x_3^{(k)}, x_3^{(k+1)}).$$

By construction we have

$$m_3(\,\cdot\,, x_3^{(k)}-) \equiv m_3(\,\cdot\,, x_3^{(k)}+) \qquad \text{and} \qquad m_3(\,\cdot\,, x_3) \xrightarrow{x_3\uparrow 1} 0,$$

thus

$$\begin{aligned}\int_{[-1,1)^2\times(0,1)} |\nabla' m_3|dx &= \sum_{k=0}^{\infty}\int_{[-1,1)^2\times(x_3^{(k)},x_3^{(k+1)})} |\nabla' m_3|dx \\ &= \sum_{k=0}^{\infty}\Delta_k 2^k \int_{[-1,1)^2\times(0,1)} |\nabla' m_3^{01}|dx\end{aligned}$$

and

$$\begin{aligned}\int_{(-1,1)^2\times(0,1)} \left||\nabla'|^{-1}\partial_3 m_3\right|^2 dx &= \sum_{k=0}^{\infty}\int_{(-1,1)^2\times(x_3^{(k)},x_3^{(k+1)})} \left||\nabla'|^{-1}\partial_3 m_3\right|^2 dx \\ &= \sum_{k=0}^{\infty}\frac{1}{\Delta_k}\left(\frac{1}{2^k}\right)^2 \int_{(-1,1)^2\times(0,1)} \left||\nabla'|^{-1}\partial_3 m_3^{01}\right|^2 dx.\end{aligned}$$

Combining these two, we see that

$$\begin{aligned}&2\int_{[-1,1)^2\times(0,1)} |\nabla' m_3|dx + \int_{(-1,1)^2\times(0,1)} \left||\nabla'|^{-1}\partial_3 m_3\right|^2 dx \\ &\le \max\left\{\sum_{k=0}^{\infty}\Delta_k 2^k, \sum_{k=0}^{\infty}\frac{1}{\Delta_k}\left(\frac{1}{2^k}\right)^2\right\}\left(2\int\limits_{[-1,1)^2\times(0,1)} |\nabla' m_3^{01}|dx + \int\limits_{(-1,1)^2\times(0,1)} \left||\nabla'|^{-1}\partial_3 m_3^{01}\right|^2 dx\right) \\ &= \max\left\{(1-\theta)\sum_{k=0}^{\infty}(2\theta)^k, \frac{1}{1-\theta}\sum_{k=0}^{\infty}\left(\frac{1}{4\theta}\right)^k\right\} \\ &\qquad \cdot\left(2\int\limits_{[-1,1)^2\times(0,1)} |\nabla' m_3^{01}|dx + \int\limits_{(-1,1)^2\times(0,1)} \left||\nabla'|^{-1}\partial_3 m_3^{01}\right|^2 dx\right),\end{aligned}$$

to ensure that the bound is finite, we need to choose $\frac{1}{4} < \theta < \frac{1}{2}$. The natural choice based on the energy scaling is $\theta = \left(\frac{1}{2}\right)^{3/2}$, but this is not of further interest here. This construction entails (43).

To finish the proof of the lemma, fix $l \ge 1$ and write

$$l = N + r \text{ with } N \in \mathbb{N} \text{ and } r \in [0,1).$$

By consecutively applying (40), (41), (42), and (43) we see

$$\begin{aligned} e^f(l) &\leq \left(\frac{N+1}{l}\right)^2 e^f(N+1) \leq \left(\frac{N+1}{N}\right)^2 e^p(N+1) \\ &\leq \left(\frac{N+1}{N}\right)^2 e^p(1) = \left(1+\frac{1}{N}\right)^2 e^p(1) < \infty \end{aligned}$$

and because $l$ was arbitrary, this entails the assertion of the lemma. □

**Lemma 9.** *In the limit $l \to \infty$ of the sharp interface model, the minimal energy among periodic configurations is no larger than that among admissible configurations with free boundary conditions, more precisely*

$$\limsup_{1 \ll l} e^p(l) \leq \liminf_{1 \ll l} e^f(l).$$

*Proof.* The main ingredients for the proof are

$$e^f(Nl_0) \geq e^f(l_0) \quad \text{for } N \in \mathbb{N} \tag{44}$$

resembling (36),

$$e^f(l) \geq \left(\frac{\tilde{l}}{l}\right)^2 e^f(\tilde{l}) \quad \text{for } \tilde{l} \leq l \tag{45}$$

analogous to (37), and

$$e^p(2l) \leq e^f(l) + \frac{4}{l}. \tag{46}$$

To verify (46), we consider an admissible pair $(m_3, h')$ for $e^f(l)$. We translate $(m_3, h')$ such that the domain is $(0, 2l)^2 \times \mathbb{R}$. We aim at the construction of an admissible pair $(\tilde{m}_3, \tilde{h}')$ for $e^p(2l)$. Unique extensions $(\tilde{m}_3, \tilde{h}')$ of $(m_3, h')$ to $\mathbb{R}^2 \times \mathbb{R}$ exist with the following properties

$$\begin{aligned} &(\tilde{m}_3, \tilde{h}') \text{ are } (-2l, 2l)^2\text{-periodic in } x', \\ &(\tilde{h}_1, \tilde{h}_2, \tilde{m}_3)(-x_1, x_2, x_3) = (\tilde{h}_1, -\tilde{h}_2, -\tilde{m}_3)(x_1, x_2, x_3), \text{ and} \\ &(\tilde{h}_1, \tilde{h}_2, \tilde{m}_3)(x_1, -x_2, x_3) = (-\tilde{h}_1, \tilde{h}_2, -\tilde{m}_3)(x_1, x_2, x_3). \end{aligned}$$

We observe

$$\begin{aligned} \left(\partial_3 \tilde{m}_3 + \partial_1 \tilde{h}_1' + \partial_2 \tilde{h}_2'\right)(-x_1, x_2, x_3) &= -\left(\partial_3 \tilde{m}_3 + \partial_1 \tilde{h}_1' + \partial_2 \tilde{h}_2'\right)(x_1, x_2, x_3), \\ \left(\partial_3 \tilde{m}_3 + \partial_1 \tilde{h}_1' + \partial_2 \tilde{h}_2'\right)(x_1, -x_2, x_3) &= -\left(\partial_3 \tilde{m}_3 + \partial_1 \tilde{h}_1' + \partial_2 \tilde{h}_2'\right)(x_1, x_2, x_3), \\ \tilde{h}_1'(0-, x_2, x_3) &= \tilde{h}_1'(0+, x_2, x_3), \\ \tilde{h}_2'(x_1, 0-, x_3) &= \tilde{h}_2'(x_1, 0+, x_3). \end{aligned}$$

Hence

$$\nabla' \cdot h' + \partial_3 m_3 = 0 \text{ in } (0, 2l)^2 \times \mathbb{R}$$

yields

$$\nabla' \cdot \tilde{h}' + \partial_3 \tilde{m}_3 = 0 \text{ in } \mathbb{R}^3.$$

Additionally

$$\tilde{m}_3^2 = \begin{cases} 1 & \text{for } x_3 \in (-1,1), \\ 0 & \text{otherwise,} \end{cases}$$

so that $(\tilde{m}_3^2, \tilde{h}')$ is admissible for $e^p(2l)$. We estimate the energy as

$$\begin{aligned} (4l)^2 e_{2l}(\tilde{m}_3, \tilde{h}') &= 2\int_{[-2l,2l)^2\times(-1,1)} |\nabla'\tilde{m}_3|dx + \int_{(-2l,2l)^2\times\mathbb{R}} |\tilde{h}'|^2 dx \\ &= 4\left(2\int_{(0,2l)^2\times(-1,1)} |\nabla'\tilde{m}_3|dx + \int_{(0,2l)^2\times\mathbb{R}} |\tilde{h}'|^2 dx\right) + 64l \\ &= 16l^2 e_l(m_3, h') + 64l. \end{aligned}$$

We incur an additional term because $\tilde{m}_3$ has a jump of height 2 at the reflection lines. Dividing by $16l^2$ yields (46).

To wrap up the proof, fix $l_0 \geq 0$ and let $l \geq l_0$ be arbitrary. Write

$$l = Nl_0 + r \text{ with } N \in \mathbb{N} \text{ and } r \in [0, l_0).$$

Having prepared our three ingredients we combine them to compute

$$\begin{aligned} e^p(2l) &\overset{(44)}{\leq} e^f(l) + \frac{4}{l} \\ &\overset{(45)}{\leq} \left(\frac{(N+1)l_0}{l}\right)^2 e^f((N+1)l_0) + \frac{4}{l} \\ &\overset{(46)}{\leq} \left(\frac{l+l_0}{l}\right)^2 e^f(l_0) + \frac{4}{l} \\ &= \left(1 + \frac{l_0}{l}\right)^2 e^f(l_0) + \frac{4}{l} \end{aligned}$$

and thus find

$$\limsup_{1\ll l} e^p(l) = \limsup_{1\ll l} e^p(2l) \leq e^f(l_0).$$

Since $l_0 > 0$ was arbitrary we have

$$\limsup_{1\ll l} e^p(l) \leq \liminf_{1\ll l} e^f(l),$$

completing the proof. □

Finally, we can collect the results on comparing the various energy estimates and deduce Theorem 2.

*Proof of Theorem 2.* Recall from Section 4 that we have to show

$$\lim_{\delta\ll\varepsilon^2\ll 1\ll l} e^p(\delta, \varepsilon, l) \in (0, \infty)$$

i.e. that the limit exists and is a strictly positive real number.

The lemmas of this section and Theorem 4 allow us to put the minimal energies for periodic and free boundary conditions and sharp and diffuse interface versions $e^p(\varepsilon,\delta,l)$, $e^f(\varepsilon,\delta,l)$, $e^p(l)$, and $e^f(l)$ in the following chain of inequalities.

$$\begin{aligned}
\limsup_{\delta\ll\varepsilon^2\ll1\ll l} e^p(\delta,\varepsilon,l) &\le \liminf_{1\ll l}\limsup_{\delta\ll\varepsilon^2\ll1} e^p(\delta,\varepsilon,l) && \text{(Lemma 7)},\\
\limsup_{\delta\ll\varepsilon^2\ll1} e^p(\delta,\varepsilon,l) &\le e^p(l) && \text{(Theorem 4, part 1)},\\
\limsup_{1\ll l} e^p(l) &\le \liminf_{1\ll l} e^f(l) && \text{(Lemma 9)},\\
e^f(l) &\le \liminf_{\delta\ll\varepsilon^2\ll1} e^f(\delta,\varepsilon,l) && \text{(Theorem 4, part 2)},\\
\limsup_{1\ll l}\liminf_{\delta\ll\varepsilon^2\ll1} e^f(\delta,\varepsilon,l) &\le \liminf_{\delta\ll\varepsilon^2\ll1\ll l} e^f(\delta,\varepsilon,l) && \text{(Lemma 6)}.
\end{aligned}$$

These combined with the trivial

$$e^f(\delta,\varepsilon,l)\le e^p(\delta,\varepsilon,l) \qquad \text{and} \qquad e^f(l)\le e^p(l)$$

imply that the limits under consideration exist and coincide for all energies. Then

$$\liminf_{l\to\infty} e^p(l) > 0 \qquad \text{(Lemma 5) and} \tag{47}$$

$$\limsup_{l\to\infty} e^f(l) < \infty \qquad \text{(Lemma 8)} \tag{48}$$

show that the limit indeed is a finite positive number. Thus the theorem is established. □

# 7 Quantification of the construction in a Modica-Mortola problem

In this largely self-contained section we provide a quantification of the construction used to show the Γ-convergence result of Modica and Mortola, [MM77] that we use in the proof of part 1 of Theorem 1. Throughout this section, we work with the half-open cubes $Q_l(x) = x + (\frac{-l}{2}, \frac{l}{2}]^n$. Our goal is to prove the following proposition:

**Proposition 4.** *For all $\alpha > 0$ a constant $C_5(\alpha,n) < \infty$ exists such that for any domain size $L > 0$, all functions $\chi : Q_L \to \{-1,1\}$ and all $\delta > 0$ there is an approximation $u : Q_L \to [-1,1]$ such that*

$$\int_{Q_L} \frac{\delta}{2}\frac{1}{1-u^2}|\nabla u|^2 + \frac{1}{2\delta}(1-u^2)dx \le (1+\alpha)\int_{Q_L}|\nabla\chi|dx$$

*and*

$$\int_{Q_L}|\chi-u| \le C_5(\alpha,n)\delta\int_{Q_L}|\nabla\chi|dx.$$

*Remark* 3. We apply the lemma in a rescaled version with $n=3$ and $\hat{x}_3 = \frac{1}{\varepsilon}x_3$ and a size in $x_3$-direction that differs from that in the other two directions. This does not affect the viability of the proposition.

First, we construct a set of finitely many characteristic functions with the property that arbitrary characteristic functions can be approximated by those from the set.

**Lemma 10.** *For all $L \in \mathbb{N}$ there is a finite set $F \subset BV(Q_L, \{-1,1\})$ with cardinality*

$$\#F \leq 2^{L^n}$$

*such that all $\chi : Q_L \to \{-1,1\}$ can be approximated by a $\tilde{\chi} \in F$ in the sense of*

$$\int_{Q_L} |\nabla\tilde{\chi}| dx \leq \int_{Q_L} |\nabla\chi| dx \text{ and} \tag{49}$$

$$\int_{Q_L} |\chi - \tilde{\chi}| dx \leq C_0(n) \int_{Q_L} |\nabla\chi| dx. \tag{50}$$

The exponent in the cardinality estimate is, of course, the volume of $Q_L$.

*Proof.* We proceed in two steps. First we approximate $\chi$ in $L^1$ by functions constant on unit cubes with an error bound proportional to the total variation, i.e. with a bound resembling (50). We then replace these initial approximation functions by minimizers of the total variation within appropriately sized closed $L^1$-neighborhoods. This ensures that the approximation satisfies (49) without making the $L^1$-error larger than twice that of the first step.

Let us decompose $Q_L$ into $L^n$ translated unit cubes $\{Q_1^k\}_{k\in\{1,\ldots,L\}^n}$ and denote by $F_0$ the set of all functions $\chi_0 : Q_L \to \{-1,1\}$ that are piecewise constant on each $Q_1^k$. Clearly $\#F_0 = 2^{L^n}$. We claim that any $\chi : Q_L \to \{-1,1\}$ can be approximated by a function $\chi_0 \in F_0$ in the sense that

$$\int_{Q_L} |\chi - \chi_0| dx \leq C_1(n) \int_{Q_L} |\nabla\chi| dx. \tag{51}$$

Indeed, let $\chi_0$ be the piecewise constant function given by

$$\chi_0\big|_{Q_1^k} = \begin{cases} 1 & \text{if } \int_{Q_1^k} \chi dx \geq 0, \\ -1 & \text{if } \int_{Q_1^k} \chi dx < 0 \end{cases}$$

for $k \in \{1,\ldots,L\}^n$. Since $Q_1^k$ has unit size, we can use the Poincaré inequality to estimate the deviation from the mean as

$$\int_{Q_1^k} |\chi - \int_{Q_1^k} \chi| \leq C_2(n) \int_{Q_1^k} |\nabla\chi| dx. \tag{52}$$

If $\int_{Q_1^k} \chi dx \geq 0$, this implies

$$\begin{aligned} \int_{Q_1^k} |\chi_0 - \chi| dx = \int_{Q_1^k} |1 - \chi| dx &= 2\mathcal{L}^n(\{x \in Q_1^k | \chi(x) = -1\}) \\ &\leq 2 \int \Big|\chi - \int_{Q_1^k} \chi\Big| dx \leq 2C_2(n) \int_{Q_1^k} |\nabla\chi| dx. \end{aligned} \tag{53}$$

The same calculation also works if $\int_{Q_1^k} \chi dx \leq 0$. Summing over $k \in \{1,\ldots,L\}^n$ we establish our claim (52) with $C_1(n) = 2C_2(n)$.

We now want to improve our choice of the approximation functions to have small total variation. To this end, consider for any $\chi_0 \in F_0$ the smallest $L^1$-neighborhood containing a good approximation. More specifically we define

$$S_P(\chi_0) := \Big\{\chi : Q_L \to \{-1,1\} \Big| \int_{Q_L} |\chi - \chi_0| dx \leq C_1(n) P\Big\}$$

and then find the minimal radius

$$P^* := P^*(\chi_0) := \inf\Big\{P\Big| \inf_{\chi\in S_P(\chi_0)} \int_{Q_L} |\nabla\chi|dx \le P\Big\}$$

that is of relevance to our approximation needs. By the standard compactness and lower semicontinuity properties of $BV$-functions both infima are, in fact, minima. We thus find $\tilde\chi \in S_{P^*}(\chi_0)$ that minimizes the total variation in the $L^1$-closed set $S_{P^*}(\chi_0)$, i.e.

$$\int_{Q_L} |\nabla\tilde\chi|dx = P^*.$$

We claim that the set

$$F = \{\tilde\chi|\chi_0 \in F_0\}$$

has the desired approximation properties. Indeed, given any $\chi : Q_L \to \{-1,1\}$ with total variation $P = \int_{Q_L} |\nabla\chi|dx$, we find by the first step a $\chi_0 \in F_0$ satisfying (51). In particular, $\chi \in S_P(\chi_0)$ and thus $P \ge P^*(\chi_0)$, which is (49). By the triangle inequality

$$S_{2P}(\tilde\chi) \supseteq S_P(\chi_0) \ni \chi,$$

in other words (50) is satisfied with $C_0(n) = 2C_1(n)$, completing the proof. □

We now use this approximation by functions from a finite set to improve the Modica-Mortola result to a uniform version, first for bounded and later for arbitrary system sizes.

**Lemma 11.** *For any system size bound $L_0 \in \mathbb{N}$ and approximation parameter $R \in \mathbb{N}$ there is a scaling coefficient $0 < \delta_0 \le 1$ such that for all $0 < \delta \le \delta_0$, all $L \le L_0$, and all functions $\chi : Q_L \to \{-1,1\}$ there is an approximating $u : Q_L \to [-1,1]$ such that the diffuse interface energy is bounded by*

$$\int_{Q_L} \frac{\delta}{2}\frac{1}{1-u^2}|\nabla u|^2 + \frac{1}{2\delta}(1-u^2)dx \le \Big(1+\frac{1}{R}\Big)\int_{Q_L}|\nabla\chi|dx \tag{54}$$

*and $u$ is close to $\chi$ in the sense that*

$$\int_{Q_L} |\chi - u|dx \le C(n)\int_{Q_L}|\nabla\chi|dx. \tag{55}$$

*Proof.* Recall from [MM77, Theorema 2] that for given fixed $L \in \mathbb{N}$ and $\delta \to 0$

$$E_\delta(u) := E_\delta(u, Q_L) := \int_{Q_L} \frac{\delta}{2}\frac{1}{1-u^2}|\nabla u|^2 + \frac{1}{2\delta}(1-u^2)dx$$

$\Gamma$-converges with respect to the $L^1(Q_L)$-topology to

$$E_0(u) := \begin{cases} \int_{Q_L}|\nabla u|dx & \text{if } u \in \{-1,+1\} \text{ a.e.,}\\ +\infty & \text{otherwise.}\end{cases}$$

With this in mind, we begin the proof. Fix an arbitrary $L_0 \in \mathbb{N}$ and $R \in \mathbb{N}$. Let us assume for the moment that $L = L_0$.

Since the set $F$ of Lemma 10 is finite, there exists a $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ and all $\tilde{\chi} \in F$, there is a $u_{\tilde{\chi}} : Q_L \to [-1,1]$ satisfying

$$E_\delta(u_{\tilde{\chi}}) \leq \Big(1 + \frac{1}{R}\Big) \int_{Q_L} |\nabla \tilde{\chi}| dx \tag{56}$$

and

$$\int_{Q_L} |\tilde{\chi} - u_{\tilde{\chi}}| \leq C_0(n) \int_{Q_L} |\nabla \tilde{\chi}| dx. \tag{57}$$

We do have the choice of using the constant in the approximation property of Lemma 10 as $C_0(n)$.

Now let $\chi : Q_L \to \{-1,1\}$ be given. According to Lemma 10, there exists $\tilde{\chi} \in F$ with (49) and (50). This allows us to estimate

$$E_\delta(u_{\tilde{\chi}}) \overset{(56)}{\leq} \Big(1 + \frac{1}{R}\Big) \int_{Q_L} |\nabla \tilde{\chi}| dx \overset{(49)}{\leq} \Big(1 + \frac{1}{R}\Big) \int_{Q_L} |\nabla \chi| dx,$$

establishing (54). To obtain (55) we compute

$$\begin{aligned}
\int_{Q_L} |\chi - u_{\tilde{\chi}}| dx \quad &\leq \quad \int_{Q_L} |\chi - \tilde{\chi}| dx + \int_{Q_L} |\tilde{\chi} - u_{\tilde{\chi}}| dx \\
&\overset{(50),(57)}{\leq} \quad C_0(n) \int_{Q_L} |\nabla \chi| dx + C_0(n) \int_{Q_L} |\nabla \tilde{\chi}| dx \\
&\overset{(49)}{\leq} \quad 2C_0(n) \int_{Q_L} |\nabla \chi| dx.
\end{aligned}$$

Thus $u_{\tilde{\chi}}$ has the properties claimed in the lemma for $\delta$ and $C(n) = 2C_0(n)$, completing the proof if $L = L_0$.

It remains to consider the case $L < L_0$. If $L \geq 1$ we rescale lengths according to

$$\hat{x} = \frac{L_0}{L} x, \qquad \hat{\delta} = \frac{L_0}{L} \delta, \qquad \hat{L} = \frac{L_0}{L} L = L_0.$$

This puts us in the case already dealt with and we obtain an approximation $\hat{u} : Q_{\hat{L}} \to [-1,1]$ for $\hat{\chi} : Q_{\hat{L}} \to \{-1,1\}$ with (54) and (55) in the new coordinates, i.e. for $\hat{\delta} \leq \delta_0$ we have

$$\int_{Q_{\hat{L}}} \frac{\hat{\delta}}{2} \frac{1}{1-\hat{u}^2} |\hat{\nabla} \hat{u}|^2 + \frac{1}{2\hat{\delta}} (1 - \hat{u}^2) d\hat{x} \leq \Big(1 + \frac{1}{R}\Big) \int_{Q_{\hat{L}}} |\hat{\nabla} \hat{\chi}| d\hat{x}$$

and

$$\int_{Q_{\hat{L}}} |\hat{\chi} - \hat{u}| d\hat{x} \leq C(n) \int_{Q_{\hat{L}}} |\hat{\nabla} \hat{\chi}| d\hat{x}.$$

Rescaling back, we notice that the constant for (55) only improves (by a factor $\frac{L}{L_0} < 1$ on the right hand side which we may drop) and (54) remains valid with $\delta = \frac{L}{L_0} \hat{\delta} \geq \frac{1}{L_0} \hat{\delta}$. As $\delta_0$ may depend on $L_0$, this is not a problem and so the claim of the lemma is established for $1 \leq L \leq L_0$ when we replace the original $\delta_0$ by $\frac{1}{L_0} \delta_0$.

Finally, we need to address the case $0 < L < 1$. Without loss of generality, we assume $\int_{Q_L} \chi dx \geq 0$. Obviously $u \equiv 1$ satisfies (54). We claim that it is also a good approximation in the sense of (55). Note that by our assumption of $\chi$ having non-negative average

$$\int_{Q_L} |\chi - u| dx = \int_{Q_L} |\chi - 1| dx = 2\mathcal{L}^n(\{x \in Q_L | x = -1\}) \leq 1,$$

so we are done if $\int_{Q_L} |\nabla \chi| dx \geq 1$. Otherwise, we can estimate similarly to (53) but this time using the Poincaré-Sobolev inequality

$$\begin{aligned}
\int_{Q_L} |\chi - u| dx &\leq \int_{Q_L} |\chi - L^{-n} \int_{Q_L} \chi| dx + \int_{Q_L} |1 - L^{-n} \int_{Q_L} \chi| dx \\
&= \int_{Q_L} |\chi - L^{-n} \int_{Q_L} \chi| dx + 2\mathcal{L}^n(\{x \in Q_L | x = -1\}) \\
&\leq 3 \int_{Q_L} |\chi - L^{-n} \int_{Q_L} \chi| dx \\
&\leq 3C_3(n) \Big( \int_{Q_L} |\nabla \chi| dx \Big)^{\frac{n}{n-1}} \\
&\leq C(n) \int_{Q_L} |\nabla \chi| dx,
\end{aligned}$$

and so (55) is verified, concluding the proof of the lemma. □

We now proceed to the core argument, where we decompose very large cubic domains into such of moderate size in order to improve the above convergence result by eliminating the dependence of the approximation length scale $\delta$ on the system size $L$.

**Lemma 12.** *For all $R > 0$ there is a $\delta > 0$ such that for any system size $\tilde{L} > 0$ and for all functions $\chi : Q_{\tilde{L}} \to \{-1, 1\}$ periodically extended to $\mathbb{R}^n$ there exists a periodic approximation $u : Q_{\tilde{L}} \to [-1, 1]$ such that*

$$\int_{Q_{\tilde{L}}} \Big( \frac{\delta}{2} \frac{1}{1-u^2} |\nabla u|^2 + \frac{1}{2\delta}(1-u^2) \Big) dx \leq \Big(1 + \frac{1}{R}\Big) \int_{Q_{\tilde{L}}} |\nabla \chi| dx, \tag{58}$$

*and*

$$\int_{Q_{\tilde{L}}} |\chi - u| dx \leq C_4(n) \int_{Q_{\tilde{L}}} |\nabla \chi| dx. \tag{59}$$

*Proof.* It is well to develop a plan before delving into the minutiae. Our basic idea is to split $Q_{\tilde{L}}$, which we think of as being very large, into cubes of a suitably chosen intermediate size $L$. Then we apply Lemma 11 to these subcubes in order to obtain approximating functions on each piece and glue together one on $Q_{\tilde{L}}$. We need to apply some care to appropriately choose the width $\Lambda$ and position of the overlap during the cutting in order to keep a lid on $|\nabla \chi|$. We also need to make a considerate choice of the region of glueing to not lose the approximation property.

Let us now fix an arbitrary $\tilde{L}$-periodic $\chi : \mathbb{R}^n \to \{-1, 1\}$. Given $\delta$ small enough, smallness depending only on $R$, our goal is to construct some $u$ satisfying

$$E_\delta(u, Q_{\tilde{L}}) \leq \Big(1 + C(n)\Big(\frac{1}{R} + \frac{\Lambda}{L} + \frac{R}{\Lambda}\Big)\Big) \int_{Q_{\tilde{L}}} |\nabla \chi| dx. \tag{60}$$

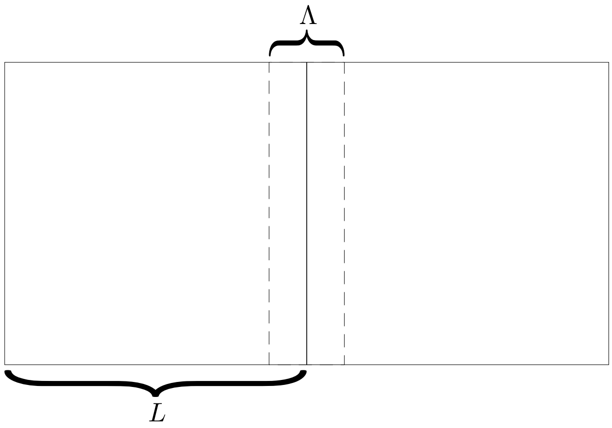

Figure 9: decomposing $Q_{\tilde{L}}$ into overlapping regions

Then we optimize the coefficient to determine $\Lambda$ and $L$ such that

$$\frac{1}{R} = \frac{\Lambda}{L} = \frac{R}{\Lambda},$$

i.e. we set

$$\Lambda = R^2 \qquad \text{and} \qquad L = R^3,$$

establishing (58) with a renaming of $R$ to compensate the constant $3C(n)$. Thus we need to achieve (60) and (59) to prove the lemma.

To begin in earnest we decompose $\mathbb{R}^n$ into cubes $\{Q^k_{L-\Lambda}\}_{k\in\mathbb{Z}^n}$ of size $L-\Lambda > 0$ and denote their centers by $x_k = (L-\Lambda)k$. For given $k \in \mathbb{Z}^n$, let $Q^k_L = Q_L(x_k)$ be the cube of size $L$ with the same center as $Q^k_{L-\Lambda}$. As alluded to above the cubes $\{Q^k_L\}_{k\in\mathbb{Z}^n}$ overlap with width $\Lambda$, see also Figure 9. Without loss of generality, we assume $R \geq 2$. This entails that the overlap width is not too large compared to the size of the $Q^k_L$, more precisely, we use that

$$\Lambda \leq \frac{L}{2}. \tag{61}$$

For convenience we also assume that $\tilde{L} = M(L-\Lambda)$ for some $M \in \mathbb{N}$ so that $M^n$ cubes $Q^k_{L-\Lambda}$ cover exactly one fundamental cell in the domain of the $\tilde{L}$-periodic functions. A variation of the rescaling used in the proof of Lemma 11 can be used to deal with nonintegral ratios greater than 1 and in the remaining case of small $\tilde{L}$ the present lemma does not claim any improvement over the previous. We remark that $\delta$, which we want to depend only on (the dimension $n$ and) the approximation quality $R$, may by above considerations also depend on the quantities $L$ and $\Lambda$ determined by $R$, a fact that shall be of use to us.

With these preparations, let us determine good areas of overlap, i.e. a good offset for the $x_k$. Using the $\tilde{L}$-periodicity we claim that there exists a translation vector $h \in \mathbb{R}^n$ such that

$$\sum_{k\in\{1,\dots,M\}^n} \int_{Q_L(x_k+h)} |\nabla\chi| dx \leq \left(\frac{L}{L-\Lambda}\right)^n \int_{Q_{\tilde{L}}} |\nabla\chi| dx. \tag{62}$$

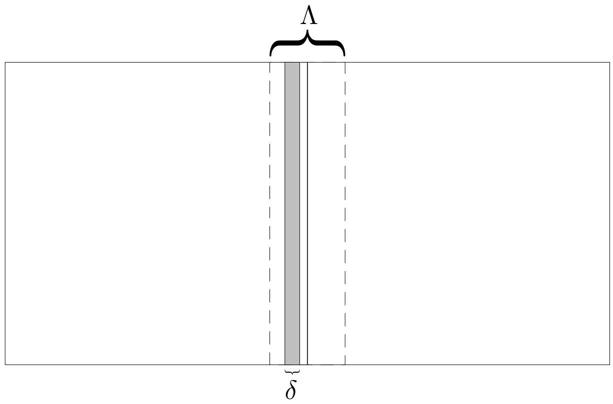


Figure 10: choosing a good set of stripes of width $\delta$ in the overlap

Indeed, we have for the average over $h' \in Q_{\tilde{L}}$

$$\begin{aligned}
&\frac{1}{\mathcal{L}^n(Q_{\tilde{L}})}\int_{Q_{\tilde{L}}}\sum_{k\in\{1,\dots,M\}^n}\int_{Q_L(x_k+h)}|\nabla\chi|dxdh' \\
&=\sum_{k\in\{1,\dots,M\}^n}\frac{\mathcal{L}^n(Q_L^k)}{\mathcal{L}^n(Q_{\tilde{L}})}\int_{Q_{\tilde{L}}}|\nabla\chi|dx \\
&=M^n\frac{L^n}{M^n(L-\Lambda)^n}\int_{Q_{\tilde{L}}}|\nabla\chi|dx,
\end{aligned}$$

and there must be an $h$ for which the integrant is bounded by the average. Without loss of generality, we assume $h=0$.

According to Lemma 11 there exists a $0<\delta=\delta(L,R)\le\frac{L}{2}$ with the property that for any $k\in\mathbb{Z}^n$ a function $u_k: Q_L^k\to[-1,1]$ exists such that

$$\int_{Q_L^k}\frac{\delta}{2}\frac{1}{1-u_k^2}|\nabla u_k|^2+\frac{1}{2\delta}(1-u_k^2)dx\le\Big(1+\frac{1}{R}\Big)\int_{Q_L^k}|\nabla\chi|dx \tag{63}$$

and

$$\int_{Q_L^k}|\chi-u_k|dx\le C(n)\int_{Q_L^k}|\nabla\chi|dx. \tag{64}$$

For given $k\in\mathbb{Z}^n$, consider $Q_{L-\Lambda+\delta/2}(x_k+h)\subset Q_L^k$ for translation vectors $h\in Q_{\Lambda-\delta/2}(0)$. In order to be able to glue functions together we are interested in the approximation quality in the boundary layer of thickness $\delta$, i.e. the set $Q_{L-\Lambda+\delta/2}(x_k+h)\setminus Q_{L-\Lambda-\delta/2}(x_k+h)$. We claim that there exists $h\in Q_{\Lambda-\delta/2}(0)$ such that

$$\sum_{k\in\{1,\dots,M\}^n}\int_{h+(Q^k_{L-\Lambda+\delta/2}\setminus Q^k_{L-\Lambda-\delta/2})}|u_k-\chi|dx\le\frac{4n\delta}{\Lambda}\sum_{k\in\{1,\dots,M\}^n}\int_{Q_L^k}|u_k-\chi|dx. \tag{65}$$

This is shown similarly to (62), this time with a one-dimensional optimization (see Figure 10): Considering stripes

$$
\begin{aligned}
S^k(h^1) := Q_L^k \cap \{x | x^1 \in x_k^1 + h^1 + \big([-(L-\Lambda)/2 - \delta/2, -(L-\Lambda)/2 + \delta/2) \\
\cup [(L-\Lambda)/2 - \delta/2, (L-\Lambda)/2 + \delta/2)\big)\}
\end{aligned}
$$

there is a $h^{1*}$ with

$$
\begin{aligned}
&\sum_{k\in\{1,\ldots,M\}^n} \int_{S^k(h^{1*})} |u_k - \chi| dx \\
&\leq \frac{1}{\Lambda - \delta/2} \int_{-\Lambda/2+\delta/4}^{\Lambda/2-\delta/4} \sum_{k\in\{1,\ldots,M\}^n} \int_{S^k(h_1)} |u_k - \chi| dx\, dh^1 \\
&\leq \frac{2\delta}{\Lambda - \delta/2} \sum_{k\in\{1,\ldots,M\}^n} \int_{Q_L^k} |u_k - \chi| dx.
\end{aligned}
$$

As noted in the beginning of the proof $\Lambda$ depends only on $R$, so we may assume $\delta \leq \Lambda/2$. Optimization for and summation over all coordinate directions yields the desired estimate (65).

Let $\{\eta_k : \mathbb{R}^n \to [0,1]\}_{k\in\mathbb{Z}^n}$ be a partition of unity subordinate to $Q_{L-\Lambda+\delta/2}(x_k+h)$. More precisely we ask that

$$
\sum_{k\in\mathbb{Z}^n} \eta_k = 1 \text{ in } \mathbb{R}^n, \tag{66}
$$

$$
\eta_k \equiv 1 \text{ on } Q_{L-\Lambda-\delta/2}(x_k+h),
$$

$$
\eta_k = 0 \text{ on } \mathbb{R}^n \setminus Q_{L-\Lambda+\delta/2}(x_k+h) \supset \mathbb{R}^n \setminus Q_L^k. \tag{67}
$$

In addition we choose $\eta_k$ such that

$$
|\nabla \eta_k|^2 \leq \frac{C}{\delta^2} \eta_k (1-\eta_k). \tag{68}
$$

Let us emphasize that this partition of unity is uniformly locally finite in the sense that for any $k$ the number of cutoff functions with support overlapping that of $\eta_k$ is bounded by a constant depending only on $n$, i.e. for all $k \in \mathbb{Z}^n$

$$
\#\{k' \in \mathbb{Z}^n | \operatorname{supp} \eta_{k'} \cap \operatorname{supp} \eta_k \neq \emptyset\} \leq C(n). \tag{69}
$$

We can now define the $\tilde{L}$-periodic function $u : \mathbb{R}^n \to [-1,1]$ as $u = \sum_{k\in\mathbb{Z}^n} \eta_k u_k$ and set out to verify (60) and later (59). We begin by noticing that

$$
\begin{aligned}
1-u^2 &= 1-\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}u_ku_{k'} \\
&= 1-\sum_{k\in\mathbb{Z}^n}\eta_k^2u_k^2-\sum_{k\in\mathbb{Z}^n}\sum_{k'\neq k}\eta_k\eta_{k'}u_ku_{k'} \\
&= 1-\sum_{k\in\mathbb{Z}^n}\eta_ku_k^2+\sum_{k\in\mathbb{Z}^n}\Big((\eta_k-\eta_k^2)u_k^2-\sum_{k'\neq k}\eta_k\eta_{k'}u_ku_{k'}\Big) \\
&\overset{(66)}{=} \sum_{k\in\mathbb{Z}^n}\eta_k(1-u_k^2)+\sum_{k\in\mathbb{Z}^n}\Big(\eta_k(1-\eta_k)u_k^2-\sum_{k'\neq k}\eta_k\eta_{k'}u_ku_{k'}\Big) \\
&\overset{(66)}{=} \sum_{k\in\mathbb{Z}^n}\eta_k(1-u_k^2)+\sum_{k\in\mathbb{Z}^n}\sum_{k'\neq k}\Big(\eta_k\eta_{k'}u_k^2-\eta_k\eta_{k'}u_ku_{k'}\Big) \\
&= \sum_{k\in\mathbb{Z}^n}\eta_k(1-u_k^2)+\sum_{k\in\mathbb{Z}^n}\sum_{k'\neq k}\eta_k\eta_{k'}u_k(u_k-u_{k'}) \\
&= \sum_{k\in\mathbb{Z}^n}\eta_k(1-u_k^2)+\frac{1}{2}\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}(u_k-u_{k'})^2 \qquad (70) \\
&\overset{(66)}{=} \frac{1}{2}\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}\left((1-u_k^2)+(1-u_{k'}^2)+(u_k-u_{k'})^2\right). \qquad (71)
\end{aligned}
$$

Using $\sum_{k\in\mathbb{Z}^n}\nabla\eta_k \overset{(66)}{=} 0$ we see that

$$
\begin{aligned}
\nabla u &= \sum_{k\in\mathbb{Z}^n}\eta_k\nabla u_k+\sum_{k\in\mathbb{Z}^n}u_k\nabla\eta_k \\
&= \sum_{k\in\mathbb{Z}^n}\eta_k\nabla u_k+\sum_{k\in\mathbb{Z}^n}\Big(u_k-\sum_{k'\in\mathbb{Z}}\eta_{k'}u_{k'}\Big)\nabla\eta_k \\
&\overset{(66)}{=} \sum_{k\in\mathbb{Z}^n}\eta_k\nabla u_k+\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}}(u_k-u_{k'})\eta_{k'}\nabla\eta_k. \qquad (72)
\end{aligned}
$$

We can thus estimate with Young's inequality

$$
|\nabla u|^2\le\Big(1+\frac{1}{R}\Big)\Big|\sum_{k\in\mathbb{Z}^n}\eta_k\nabla u_k\Big|^2+(1+R)\Big|\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}}(u_k-u_{k'})\eta_{k'}\nabla\eta_k\Big|^2.
$$

Combining this with (70) and (71) we get

$$
\begin{aligned}
\frac{1}{1-u^2}|\nabla u|^2 &\le \Big(1+\frac{1}{R}\Big)\frac{\big|\sum_{k\in\mathbb{Z}^n}\eta_k\nabla u_k\big|^2}{\sum_{k\in\mathbb{Z}^n}\eta_k(1-u_k^2)} \\
&\quad+(1+R)\frac{\big|\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}}(u_k-u_{k'})\eta_{k'}\nabla\eta_k\big|^2}{\frac{1}{2}\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}\left((1-u_k^2)+(1-u_{k'}^2)+(u_k-u_{k'})^2\right)}.
\end{aligned}
$$

We use the convexity of $(v,g)\mapsto\frac{1}{v}|g|^2$ on $(0,\infty)\times\mathbb{R}^n$ to estimate by pulling the (locally finite) summation in the first term out of the fraction and obtain

$$
\begin{aligned}
\frac{1}{1-u^2}|\nabla u|^2 &\le \Big(1+\frac{1}{R}\Big)\sum_{k\in\mathbb{Z}^n}\eta_k\frac{|\nabla u_k|^2}{1-u_k^2} \\
&\quad+(1+R)\frac{\big|\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}}(u_k-u_{k'})\eta_{k'}\nabla\eta_k\big|^2}{\frac{1}{2}\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}\left((1-u_k^2)+(1-u_{k'}^2)+(u_k-u_{k'})^2\right)}.
\end{aligned}
$$

In combination with (70), this entails

$$
\begin{aligned}
&\frac{\delta}{2}\frac{1}{1-u^2}|\nabla u|^2+\frac{1}{2\delta}(1-u^2)\\
&\le \Big(1+\frac{1}{R}\Big)\sum_{k\in\mathbb{Z}^n}\eta_k\Big(\frac{\delta}{2}\frac{|\nabla u_k|^2}{1-u_k^2}+\frac{1}{2\delta}(1-u_k^2)\Big)\\
&\quad+(1+R)\frac{\delta\big|\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}}(u_k-u_{k'})\eta_{k'}\nabla\eta_k\big|^2}{\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}\left((1-u_k^2)+(1-u_{k'}^2)+(u_k-u_{k'})^2\right)}\\
&\quad+\frac{1}{4\delta}\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}(u_k-u_{k'})^2\\
&=: S_1+S_2+S_3. 
\end{aligned}\tag{73}
$$

We address the terms on the right hand side separately. Starting with $S_1$ we write

$$
\begin{aligned}
\int_{Q_{\tilde L}}S_1dx &= \Big(1+\frac{1}{R}\Big)\int_{Q_{\tilde L}}\sum_{k\in\mathbb{Z}^n}\eta_k\Big(\frac{\delta}{2}\frac{|\nabla u_k|^2}{1-u_k^2}+\frac{1}{2\delta}(1-u_k^2)\Big)dx\\
&\overset{(67)}{\le} \Big(1+\frac{1}{R}\Big)\sum_{k\in\{1,\dots,M\}^n}E_\delta\big(u_k,Q_{L-\Lambda+\delta/2}(x_k+h)\big)\\
&\le \Big(1+\frac{1}{R}\Big)\sum_{k\in\{1,\dots,M\}^n}E_\delta\big(u_k,Q_L^k\big)\\
&\overset{(63)}{\le} \Big(1+\frac{1}{R}\Big)^2\sum_{k\in\{1,\dots,M\}^n}\int_{Q_L^k}|\nabla\chi|dx\\
&\overset{(62)}{\le} \Big(1+\frac{1}{R}\Big)^2\Big(\frac{L}{L-\Lambda}\Big)^n\int_{Q_{\tilde L}}|\nabla\chi|dx.
\end{aligned}\tag{74}
$$

We proceed to estimate $S_2+S_3$ at any point $x\in\mathbb{R}^n$. To this end, assume without loss of generality $\chi(x)=1$ and let $J=J(x)=\{k\in\mathbb{Z}^n|x\in\operatorname{supp}\eta_k\}$. Using the local finiteness (69) and $R\ge 1$ we see

$$
\begin{aligned}
&S_2+S_3\\
&= (1+R)\frac{\delta\big|\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}}(u_k-u_{k'})\eta_{k'}\nabla\eta_k\big|^2}{\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}\left((1-u_k^2)+(1-u_{k'}^2)+(u_k-u_{k'})^2\right)}\\
&\quad+\frac{1}{4\delta}\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}(u_k-u_{k'})^2\\
&\le C(n)R\frac{\delta\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}}(u_k-u_{k'})^2\eta_{k'}^2|\nabla\eta_k|^2}{\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}\left((1-u_k^2)+(1-u_{k'}^2)+(u_k-u_{k'})^2\right)}\\
&\quad+\frac{1}{4\delta}\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}(u_k-u_{k'})^2\\
&\overset{(68)}{\le} \frac{1}{\delta}\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}(u_k-u_{k'})^2\\
&\qquad\Big(\frac{1}{4}+\frac{C(n)R\eta_{k'}(1-\eta_k)}{\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}\left((1-u_k^2)+(1-u_{k'}^2)+(u_k-u_{k'})^2\right)}\Big)\\
&\overset{(\eta_k\le 1)}{\le} \frac{1}{\delta}\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}(u_k-u_{k'})^2\Big(\frac{1}{4}+\frac{C(n)R}{(1-u_k^2)+(1-u_{k'}^2)+(u_k-u_{k'})^2}\Big).
\end{aligned}\tag{75}
$$

We now claim

$$\frac{(u_k - u_{k'})^2}{(1-u_k^2) + (1-u_{k'}^2) + (u_k - u_{k'})^2} \le |u_k - u_{k'}| + (1-u_k) + (1-u_{k'}). \tag{76}$$

If $u_k \le 0$ or $u_{k'} \le 0$, the left hand side smaller than 1 while the right hand side is larger, so that the inequality is trivial in this case. For $u_k \ge 0$ and $u_{k'} \ge 0$ we start with the elementary observation

$$(a-b)^2 \le |a-b||a+b|,$$

which, for $a, b \ge 0$ is equivalent to

$$\frac{(a-b)^2}{a+b} \le |a-b|.$$

Plugging in $a = 1 - u_k$ and $b = 1 - u_{k'}$, this becomes

$$\frac{(u_k - u_{k'})^2}{(1-u_k) + (1-u_{k'})} \le |u_k - u_{k'}|,$$

which, by $1 - u^2 = (1-u)(1+u) \ge 1 - u$ for $u \ge 0$ and adding non-negative terms to the denominator and right hand side implies

$$\begin{aligned} &\frac{(u_k - u_{k'})^2}{(1-u_k^2) + (1-u_{k'}^2) + (u_k - u_{k'})^2} \\ &\qquad \le \frac{(u_k - u_{k'})^2}{(1-u_k) + (1-u_{k'})} \le |u_k - u_{k'}| \le |u_k - u_{k'}| + (1-u_k) + (1-u_{k'}). \end{aligned}$$

Thus (76) is established.

We can now continue with our estimation (75), we start with using $(u_1 - u_2)^2 \le 2|u_1 - u_2|$

$$\begin{aligned} &S_2 + S_3 \\ &\overset{(75)}{\le} \frac{1}{\delta} \sum_{k\in\mathbb{Z}^n} \sum_{k'\in\mathbb{Z}^n} \eta_k \eta_{k'} (u_k - u_{k'})^2 \Big( \frac{1}{4} + \frac{C(n)R}{(1-u_k^2) + (1-u_{k'}^2) + (u_k - u_{k'})^2} \Big) \\ &\le \frac{1}{\delta} \sum_{k\in\mathbb{Z}^n} \sum_{k'\in\mathbb{Z}^n} \eta_k \eta_{k'} \Big( \frac{1}{2} |u_k - u_{k'}| + \frac{C(n)R(u_k - u_{k'})^2}{(1-u_k^2) + (1-u_{k'}^2) + (u_k - u_{k'})^2} \Big) \\ &\overset{(76)}{\le} \frac{C(n)R}{\delta} \sum_{k\in\mathbb{Z}^n} \sum_{k'\neq k} \eta_k \eta_{k'} (|u_k - u_{k'}| + (1-u_k) + (1-u_{k'})) \\ &\le \frac{C(n)R}{\delta} \sum_{k\in\mathbb{Z}^n} \sum_{k'\neq k} \eta_k \eta_{k'} (|\chi - u_k| + |\chi - u_{k'}|) \\ &\overset{(66)}{\le} \frac{C(n)R}{\delta} \sum_{k\in\mathbb{Z}^n} \eta_k (1-\eta_k) |\chi - u_k|, \end{aligned} \tag{77}$$

in the last two estimates we use the triangle inequality and our assumption $\chi(x) = 1$.

Using our choice of the boundary layer we estimate the integral over $S_2 + S_3$ (which are supported only on the boundary layers) as

$$\begin{aligned}
&\int_{Q_{\tilde L}} S_2+S_3dx\\
&\overset{(77)}{\le} \quad \frac{C(n)R}{\delta}\sum_{k\in\{1,\ldots,M\}^n}\int_{Q_{\tilde L}}\eta_k(1-\eta_k)|\chi-u_k|dx\\
&\overset{(67)}{\le} \quad \frac{C(n)R}{\delta}\sum_{k\in\{1,\ldots,M\}^n}\int_{Q_L^k}\eta_k(1-\eta_k)|\chi-u_k|dx\\
&\overset{(65),(67)}{\le} \quad \frac{C(n)R}{\Lambda}\sum_{k\in\{1,\ldots,M\}^n}\int_{Q_L^k}|\chi-u_k|dx\\
&\overset{(64)}{\le} \quad \frac{C(n)R}{\Lambda}\sum_{k\in\{1,\ldots,M\}^n}\int_{Q_L^k}|\nabla\chi|dx\\
&\overset{(62),(61)}{\le} \quad \frac{C(n)R}{\Lambda}\int_{Q_{\tilde L}}|\nabla\chi|dx.
\end{aligned} \tag{78}$$

Combining (73), (74), and (78), we see that

$$\begin{aligned}
&\int_{Q_{\tilde L}} \frac{\delta}{2}\frac{1}{1-u^2}|\nabla u|^2+\frac{1}{2\delta}(1-u^2)dx\\
&\overset{(73)}{\le} \quad \int_{Q_{\tilde L}} S_1+S_2+S_3dx\\
&\overset{(74),(78)}{\le} \quad \Big(\Big(1+\frac{1}{R}\Big)^2\Big(\frac{L}{L-\Lambda}\Big)^n+\frac{C(n)R}{\Lambda}\Big)\int_{Q_{\tilde L}}|\nabla\chi|dx\\
&\overset{(61),(R\ge 1)}{\le} \quad \Big(1+C(n)\Big(\frac{1}{R}+\frac{\Lambda}{L}+\frac{R}{\Lambda}\Big)\Big)\int_{Q_{\tilde L}}|\nabla\chi|dx.
\end{aligned}$$

But this is (60), which we know from above to imply (58). To complete the proof of the lemma we need to verify the approximation property (59). By definition of $u$

$$\begin{aligned}
\int_{Q_{\tilde L}}|\chi-u|dx \quad &= \quad \int_{Q_{\tilde L}}\Big|\chi-\sum_{k\in\mathbb{Z}^n}\eta_k u_k\Big|dx\\
&\overset{(66)}{=} \quad \int_{Q_{\tilde L}}\Big|\sum_{k\in\mathbb{Z}^n}\eta_k(\chi-u_k)\Big|dx\\
&\overset{(67)}{\le} \quad \sum_{k\in\{1,\ldots,M\}^n}\int_{Q_L^k}\Big|\chi-u_k\Big|dx\\
&\overset{(64)}{\le} \quad C(n)\sum_{k\in\{1,\ldots,M\}^n}\int_{Q_L^k}|\nabla\chi|dx\\
&\overset{(62)}{\le} \quad C(n)\Big(\frac{L}{L-\Lambda}\Big)^n\int_{Q_{\tilde L}}|\nabla\chi|dx\\
&\overset{(61)}{\le} \quad C(n)\int_{Q_{\tilde L}}|\nabla\chi|dx,
\end{aligned}$$

which is (59), the missing piece in the proof of our lemma. □

Finally, we prove Proposition 4.

*Proof of Proposition 4.* Let $\alpha > 0$ be given and set $R = \frac{1}{\alpha}$. Denote by $\hat\delta = \hat\delta(R)$ the parameter of Lemma 12 and let $C_5(n,R) = \frac{1}{\hat\delta}C_4(n)$.

We rescale the lengths according to

$$x = \frac{\delta}{\hat\delta}\hat x, \qquad L = \frac{\delta}{\hat\delta}\hat L.$$

According to Lemma 12, there exists $\hat u : Q_{\hat L} \to [-1,1]$ such that

$$\int_{Q_{\hat L}} \left(\frac{\hat\delta}{2}\frac{1}{1-\hat u^2}|\hat\nabla \hat u|^2 + \frac{1}{2\hat\delta}(1-\hat u^2)\right) d\hat x \le \left(1+\frac{1}{R}\right)\int_{Q_{\hat L}} |\hat\nabla\hat\chi| d\hat x,$$

and

$$\int_{Q_{\hat L}} |\hat\chi - \hat u| d\hat x \le C_4(n)\int_{Q_{\hat L}} |\hat\nabla \hat\chi| d\hat x.$$

Rescaling back this gives $u : Q_L \to [-1,1]$ such that

$$\int_{Q_L} \left(\frac{\hat\delta}{2}\frac{1}{1-u^2}\left(\frac{\delta}{\hat\delta}\right)^2|\nabla u|^2 + \frac{1}{2\hat\delta}(1-u^2)\right) dx \le \left(1+\frac{1}{R}\right)\frac{\delta}{\hat\delta}\int_{Q_L} |\nabla\chi| dx,$$

and

$$\int_{Q_L} |\chi - u| dx \le C_4(n)\frac{\delta}{\hat\delta}\int_{Q_L} |\nabla\chi| dx$$

as desired. □

# 8 Local behavior of the energy in minimizers

We now prove Theorem 3. In the following we refer to the sharp-interface energy

$$E_{m_3,h'}(x_0, l_{x'}, l_{x_3}) := 2\int_{x_0+([-l_{x'},l_{x'})^2\times(0,l_{x_3}))} |\nabla' m_3| dx + \int_{x_0+((-l_{x'},l_{x'})^2\times(0,l_{x_3}))} |h'|^2 dx$$

as the energy for configurations with

$$\partial_3 m_3 + \nabla'\cdot h' = 0$$

and appropriate boundary conditions.

The lower bound of the theorem is essentially an application of Lemma 5.

**Lemma 13.** *There is a universal constant $C$ such that any energy-minimizing configuration $m_3$, $h'$ defined on $(-l,l)^2\times(0,2)$ and $(-l,l)^2$-periodic in $x'$ with $m_3 \rightharpoonup 0$ weakly as $x_3 \to \{0,2\}$ has the following property: For any $x_0' \in (-l,l)^2$ and any $l \ge l_{x'} \ge C l_{x_3}^{2/3}$*

$$E_{m_3,h'}(x_0', l_{x'}, l_{x_3}) := 2\int_{(x_0'+(-l_{x'},l_{x'})^2)\times(0,l_{x_3})} |\nabla' m_3| dx + \int_{(x_0'+[-l_{x'},l_{x'})^2)\times(0,l_{x_3})} |h'|^2 dx \ge C l_{x_3}^{1/3} l_{x'}^2.$$

*The constants are universal in the sense that they are independent of $l$, $x_0$, $l_{x'}$, and $l_{x_3}$.*

*Proof.* Without loss of generality we assume $x'_0 = 0$. We reflect the magnetization evenly and field oddly at $x_3 = l_{x_3}$. After another set of reflections at $x_1 = \pm l_{x'}$ and $x_2 = \pm l_{x'}$, even in the $h'$-component in the direction of extension, odd in $m_3$ and the other $h'$-component, as detailed in the proof of Lemma 9 and rescaling $x_3$ by $l_{x_3}^{-1}$ and $x'$ by $l_{x_3}^{-2/3}$ as described at the end of Section 4 we obtain an $(-2l_{x_3}^{-2/3}l_{x'}, 2l_{x_3}^{-2/3}l_{x'})$-periodic configuration with $m_3 \rightharpoonup 0$ as $x_3 \to \{0, 2\}$ with energy

$$8l_{x_3}^{-5/3}E_{m_3,h'}(x'_0, l_{x'}, l_{x_3}) + Cl_{x_3}^{-2/3}l_{x'}.$$

By Lemma 5 any $(-2l_{x_3}^{-2/3}l_{x'}, 2l_{x_3}^{-2/3}l_{x'})^2$-periodic configuration on $(-2l_{x_3}^{-2/3}l_{x'}, 2l_{x_3}^{-2/3}l_{x'})^2 \times (0, 2)$ has energy bounded from below by $\frac{1}{C}l_{x_3}^{-4/3}l_{x'}^2$. Thus

$$E_{m_3,h'}(x'_0, l_{x'}, l_{x_3}) \geq \frac{1}{C}l_{x_3}^{1/3}l_{x'}^2 - Cl_{x_3}l_{x'}$$

which is the claim of the lemma when we bound the second term on the right hand side by one half of the first for $l_{x'} \geq 2C^2 l_{x_3}^{2/3}$. $\square$

Interestingly, but not of relevance here, the analogue of the equipartition of energy result [KM94, Lemma 2.6] implies that cutting out a sample piece around the center in $x_3$-direction (i.e. taking a periodic minimizer and performing only the first, vertical reflection described above) does not yield a minimizer: The slicewise field energy of minimizers converges to 0 at the center for minimizers, it does not for the constructed comparison function.

The upper bound is one of the claims of Theorem 5 proved in the remainder of this section.

## 8.1 Upper bound for the energy in subdomains

The local upper estimates for the energy are derived in two steps. We consider a fixed minimizing configuration $(m_3, h')$. We drop the (now fixed) $m_3$ and $h'$ (unless that would lead to confusion) and write $E$ as a function of the extension of the cuboid in which we integrate the energy density, i.e.

$$E(l_{x'}, l_{x_3}) := 2\int_{[-l_{x'},l_{x'})^2\times(0,l_{x_3})} |\nabla' m_3|dx + \int_{(-l_{x'},l_{x'})^2\times(0,l_{x_3})} |h'|^2 dx.$$

In fact we modify the energy by subtracting the $x_3$-average $\bar{h}'(x') = \int_0^{l_{x_3}} h' dx_3$ of $h'$ and consider

$$\tilde{E}(l_{x'}, l_{x_3}) := 2\int_{[-l_{x'},l_{x'})^2\times(0,l_{x_3})} |\nabla' m_3|dx + \int_{(-l_{x'},l_{x'})^2\times(0,l_{x_3})} |h' - \bar{h}'|^2 dx,$$

see below for more discussion.

Before we begin we make a first observation concerning the two energies $E$ and $\tilde{E}$.

**Lemma 14.** *Given top and bottom magnetization as functions*

$$m_3^{\mathrm{T}}, m_3^{\mathrm{B}} : (-l_{x'}, l_{x'})^2 \to [-1, 1],$$

*a field flux*

$$f : \partial(-l_{x'}, l_{x'})^2 \times (0, l_{x_3}) \to \mathbb{R}$$

*across the sides, and curl-free cumulated fields $H'_{\rm T}$, $H'_{\rm B} : (-l_{x'}, l_{x'})^2 \to \mathbb{R}^2$ at top and bottom with*

$$\begin{aligned} -\nabla' \cdot H'_{\rm T} &= m_3^{\rm T} \textit{ in } (-l_{x'}, l_{x'})^2, \\ -\nabla' \cdot H'_{\rm B} &= m_3^{\rm B} \textit{ in } (-l_{x'}, l_{x'})^2, \\ \nu' \cdot (H'_{\rm T} - H'_{\rm B})(x') &= \int_0^{l_{x_3}} f(x', x_3) dx_3 \textit{ for } \mathcal{H}^1\textit{-a.e. } x' \in \partial(-l_{x'}, l_{x'})^2 \end{aligned}$$

*let $m_3 : (-l_{x'}, l_{x'})^2 \times (0, l_{x_3}) \to \{+1, -1\}$ and $h' : (-l_{x'}, l_{x'})^2 \times (0, l_{x_3}) \to \mathbb{R}^2$ be an energy-minimizing configuration among all $(m_3, h')$ such that*

$$\begin{aligned} \partial_3 m_3 + \nabla' \cdot h' &= 0 \textit{ distributionally in } (-l_{x'}, l_{x'})^2 \times (0, l_{x_3}), \\ m_3 &\rightharpoonup m_3^{\rm B} \textit{ weakly as } x_3 \to 0, \\ m_3 &\rightharpoonup m_3^{\rm T} \textit{ weakly as } x_3 \to l_{x_3}, \\ h' \cdot \nu' &= f \textit{ on } \partial(-l_{x'}, l_{x'})^2 \times (0, l_{x_3}). \end{aligned}$$

*Then*

$$\int_0^{l_{x_3}} h'(x', x_3) dx_3 = (H'_{\rm T} - H'_{\rm B})(x') \textit{ for a.e. } x' \in (-l_{x'}, l_{x'})^2. \tag{79}$$

*Proof.* When we fix the magnetization $m_3$ the field $h'$ is a minimizer, i.e.

$$\int_{(-l_{x'},l_{x'})^2\times(0,l_{x_3})} |h'|^2 dx = \min_{\tilde{h}'\in\mathcal{A}} \int_{(-l_{x'},l_{x'})^2\times(0,l_{x_3})} |\tilde{h}'|^2 dx$$

with the admissible class

$$\mathcal{A} = \big\{\tilde{h}' \big| \tilde{h}' \cdot \nu' = f \text{ on } \partial(-l_{x'}, l_{x'})^2 \times (0, l_{x_3}),\ \nabla' \cdot \tilde{h}' + \partial_3 m_3 = 0 \text{ in } (-l_{x'}, l_{x'})^2 \times (0, l_{x_3})\big\}.$$

By decomposing $h'$ orthogonally into $x_3$-average and $x_3$-oscillation we can split the minimization

$$\min_{\tilde{h}'\in\mathcal{A}} \int_{(-l_{x'},l_{x'})^2\times(0,l_{x_3})} |\tilde{h}'|^2 dx = \min_{\tilde{h}'_0\in\mathcal{A}_0} \int_{(-l_{x'},l_{x'})^2\times(0,l_{x_3})} |\tilde{h}'_0|^2 dx + \min_{\tilde{h}'_1\in\mathcal{A}_1} l_{x_3} \int_{(-l_{x'},l_{x'})^2} |\tilde{h}'_1|^2 dx'.$$

with the admissible class decomposed into

$$\begin{aligned} \mathcal{A}_0 &= \big\{\tilde{h}_0 \big| \tilde{h}_0 \cdot \nu' = f - l_{x_3}^{-1}(H_{\rm T} - H_{\rm B}) \cdot \nu' \text{ on } \partial(-l_{x'}, l_{x'})^2 \times (0, l_{x_3}), \\ &\qquad \nabla' \cdot \tilde{h}'_0 + \partial_3 m = 0 \text{ in } (-l_{x'}, l_{x'})^2 \times (0, l_{x_3}), \\ &\qquad \int_0^{l_{x_3}} \tilde{h}_0(x', \xi_3) d\xi_3 = 0 \text{ for a.e. } x' \in (-l_{x'}, l_{x'})^2\big\}, \\ \mathcal{A}_1 &= \big\{\tilde{h}_1 \big| \tilde{h}_1 \cdot \nu' = l_{x_3}^{-1}(H_{\rm T} - H_{\rm B}) \cdot \nu' \text{ on } \partial(-l_{x'}, l_{x'})^2 \times (0, l_{x_3}), \\ &\qquad \nabla' \cdot \tilde{h}_1 = -l_{x_3}^{-1}(m^{\rm T} - m^{\rm B}) \text{ in } (-l_{x'}, l_{x'})^2\big\}. \end{aligned}$$

As $l_{x_3}^{-1}(H_{\rm T} - H_{\rm B})$ is curl-free by assumption, it also solves the minimization problem in $\mathcal{A}_1$. By uniqueness of the minimizer of the strictly convex minimization problem, we have $l_{x_3}^{-1}(H_{\rm T} - H_{\rm B}) = h_1$, as claimed. □

As the first step, in an inner iteration we consider subdomains $Q(l_{x'}, l_{x_3}) := (-l_{x'}, l_{x'})^2 \times (0, l_{x_3})$ for fixed $l_{x_3}$ and varying (but not too small in a sense to be made precise) $l_{x'}$. Quite literally, our starting point are large horizontal cubes, $l_{x'} = l$. For these the bound $\tilde{E}(l_{x'}, l_{x_3}) \leq C_E l_{x_3}^{1/3} l_{x'}^2$ is essentially established

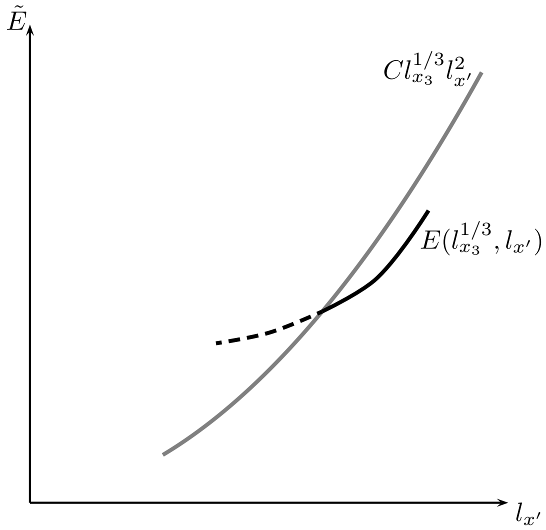

Figure 11: The ODE argument in a nutshell: if $E(l_{x'}, l_{x_3})$ intersects with $Cl_{x_3}^{1/3}l_{x'}^2$ it has smaller derivative there

in [CK98] in a setting similar in Lemma 8, we present a slight variation below. We now want to extend the control of the energy as $l_{x'}$ decreases.

Our argument and presentation resembles the "ODE argument" of [ACO06, Lemma 3.5], the one in [Con00] is very similar.

Imagine that $l_{x'} \mapsto \tilde{E}(l_{x_3}, l_{x'})$ were smooth. If $\tilde{E}(l_{x_3}, l_{x'}) > C_E l_{x_3}^{1/3} l_{x'}^2$ for some $l_{x'}$ then there is a largest horizontal length $l_*$ such that $\tilde{E}(l_*) \geq C_E l_*^2$. This implies that at $l_*$ the energy must grow more slowly than $C_E l_{x_3}^{1/3} l_{x'}^2$, i.e. $\tilde{E}'(l_*) \leq (C_E l_{x_3}^{1/3} l_{x'}^2)'\big|_{l_{x'}=l_*} = 2C_E l_{x_3}^{1/3} l_*$. Combining these two inequalities with a differential inequality relating $\tilde{E}'$ and $\tilde{E}$ we can obtain an estimate for $l_*$.

The differential inequality in [ACO06] takes (in two dimensions) the form

$$E(l_{x'}) \leq C(l_{x'}^2 + (E'(l_{x'}))^{4/3}) \tag{80}$$

to allow the conclusion that $l_*$ is bounded by a constant independent of the system size $l$. This inner iteration with the ODE argument is done rigorously in Section 8.2.

As we want to construct comparison configurations for configurations with prescribed boundary values, we need to introduce the cumulated field with more precision. We consider a cuboid domain and prescribe a (relaxed) magnetization $m_3(x) \in [-1, 1]$ to be weakly assumed at the top and bottom boundary and a field flux $\nu \cdot h'$ across the vertical boundaries. For convenience, we move the domain to $(-l_{x'}, l_{x'})^2 \times (0, l_{x_3})$. The most important case is a cuboid close to the sample boundary $\{x_3 = 0\}$ with $m_3(x', 0) = 0$ weakly. If $x_3 = 0$ is a sample boundary where the magnetization vanishes weakly, we set

$$H'(x) = \int_0^{x_3} h'(x', \xi_3) d\xi_3.$$

We do, however, need a slight generalization, so for

$$m_3^{\mathrm{B}} \in [-1, 1]$$

and possibly some prescribed flux at the boundary there is an $L^2$-minimal $H'_{\mathrm{B}}$ such that

$$-\nabla' \cdot H'_{\mathrm{B}} = m_3^{\mathrm{B}}$$

and define for a configuration with $m_3 \rightharpoonup m_3^{\mathrm{B}}$ weakly as $x_3 \to 0$ the cumulated field as

$$H'(x) = H'_{\mathrm{B}}(x') + \int_0^{x_3} h'(x', \xi_3) d\xi_3.$$

Loosely speaking, $\sup |H'|$ bounds the domain width. The field $H'$ is the equivalent of the scalar function $u$ in the functional of Kohn and Müller in [KM94]. We sometimes speak of $\int |H'|^2 dx'$ as the energy of $H'$.

In the second, outer iteration we use the local bound to show decay of the cumulated field $H'$ towards the boundary and derive the desired energy bounds. The course of our arguments is in this step modeled after [Con00]. Here we consider the difference to the linear interpolation again. With the aid of the local bounds from the inner step we can estimate the distance to the linear interpolation. Crucially, we can do so in a way that has a less-than-linear dependence on the constant relating $|H'|$ to its expected scaling. Focusing on the center of the $x_3$-interval we can then use this fact and that decay of the linear interpolation (trivially exponent 1) decays better than the expected scaling (exponent $\frac{2}{3}$) to absorb the constants from our estimates and get the desired scaling result for $H'$. This is done in Section 8.3.

## 8.2 Localization of the energy estimate w.r.t. the horizontal directions

In this subsection we provide the differential inequality and then make the ODE argument introduced above precise. As usual constants such as the generic $C$ and those named after lemmas and the proposition, e.g. $C_{L1}$, $C_{P1}$, are allowed to depend on each other (but not cyclically). However, in this section they crucially do not depend on $C_{H'}$. Instead the dependence of estimates on $C_{H'}$ is always explicitly tracked. This is important because we choose $C_{H'}$ to suit our needs based on the other constants in the next subsection.

**Proposition 5.** *Given top and bottom magnetization as functions*

$$m_3^{\mathrm{T}}, m_3^{\mathrm{B}} : (-l_{x'}, l_{x'})^2 \to [-1, 1],$$

*a field flux*

$$f : \partial(-l_{x'}, l_{x'})^2 \times (0, l_{x_3}) \to \mathbb{R}$$

*across the sides, and cumulated fields $H'_{\mathrm{T}}$, $H'_{\mathrm{B}} : (-l_{x'}, l_{x'})^2 \to \mathbb{R}^2$ at top and bottom with*

$$\begin{aligned} -\nabla' \cdot H'_{\mathrm{T}} &= m_3^{\mathrm{T}} \textit{ in } (-l_{x'}, l_{x'})^2, \\ -\nabla' \cdot H'_{\mathrm{B}} &= m_3^{\mathrm{B}} \textit{ in } (-l_{x'}, l_{x'})^2, \\ \nu' \cdot (H'_{\mathrm{T}} - H'_{\mathrm{B}})(x') &= \int_0^{l_{x_3}} f(x', x_3) dx_3 \textit{ for } \mathcal{H}^1\textit{-a.e. } x' \in \partial(-l_{x'}, l_{x'})^2 \end{aligned}$$

*let $m_3 : (-l_{x'}, l_{x'})^2 \times (0, l_{x_3}) \to \{+1, -1\}$ and $h' : (-l_{x'}, l_{x'})^2 \times (0, l_{x_3}) \to \mathbb{R}^2$ be an energy-minimizing configuration among all $(m_3, h')$ such that*

$$\begin{aligned} \partial_3 m_3 + \nabla' \cdot h' &= 0 \textit{ distributionally in } (-l_{x'}, l_{x'})^2 \times (0, l_{x_3}), \\ m_3 &\rightharpoonup m_3^{\mathrm{B}} \textit{ weakly as } x_3 \to 0, \\ m_3 &\rightharpoonup m_3^{\mathrm{T}} \textit{ weakly as } x_3 \to l_{x_3}, \\ h' \cdot \nu' &= f \textit{ on } \partial(-l_{x'}, l_{x'})^2 \times (0, l_{x_3}). \end{aligned}$$

*Given a constant $C_{H'} \geq 1$ let $H'$, $f$, and $l_{x'}$ satisfy*

$$\begin{aligned} \sup|H'_{\mathrm{T}}|, \sup|H'_{\mathrm{B}}| &\leq C_{H'} l_{x_3}^{2/3}, && (81)\\ l_{x'} &\geq \frac{16}{3} C_{H'} l_{x_3}^{2/3}, \\ \int_{\partial(-l_{x'},l_{x'})^2\times(0,l_{x_3})} |f - \frac{1}{l_{x_3}}\nu'\cdot(H'_{\mathrm{T}} - H'_{\mathrm{B}})|^2 dx &\leq 2^{-9} l_{x_3} l_{x'}^3. && (82) \end{aligned}$$

*Then the energy of $(m_3, h')$ in $(-l_{x'}, l_{x'})^2 \times (0, l_{x_3})$ is bounded by*

$$\begin{aligned} \tilde{E}(l_{x'}, l_{x_3}) &:= 2\int_{[-l_{x'},l_{x'})^2\times(0,l_{x_3})} |\nabla' m_3| dx + \int_{(-l_{x'},l_{x'})^2\times(0,l_{x_3})} |h' - \frac{1}{l_{x_3}}(H'_{\mathrm{T}} - H'_{\mathrm{B}})|^2 dx \\ &\leq C_{P5} l_{x_3}^{1/3} l_{x'}^2 + C_{P5} C_{H'} l_{x_3}^{2/3} \int_{\partial(-l_{x'},l_{x'})^2\times(0,1)} |f - \frac{1}{l_{x_3}}\nu'\cdot(H'_{\mathrm{T}} - H'_{\mathrm{B}})|^2 dx' \\ &\quad + C_{P5} l_{x_3}^{1/3} \left( \int_{\partial(-l_{x'},l_{x'})^2\times(0,l_{x_3})} |f - \frac{1}{l_{x_3}}\nu'\cdot(H'_{\mathrm{T}} - H'_{\mathrm{B}})|^2 dx' \right)^{4/3}. \end{aligned}$$

*If $H'_{\mathrm{T}} - H'_{\mathrm{B}}$ is curl-free we may rewrite this as*

$$\begin{aligned} E(l_{x'}, l_{x_3}) &:= 2\int_{[-l_{x'},l_{x'})^2\times(0,l_{x_3})} |\nabla' m_3| dx + \int_{(-l_{x'},l_{x'})^2\times(0,l_{x_3})} |h'|^2 dx \\ &\leq C_{P5} l_{x_3}^{1/3} l_{x'}^2 + C_{P5} C_{H'} l_{x_3}^{2/3} \int_{\partial(-l_{x'},l_{x'})^2\times(0,1)} |f - \frac{1}{l_{x_3}}\nu'\cdot(H'_{\mathrm{T}} - H'_{\mathrm{B}})|^2 dx' \\ &\quad + C_{P5} l_{x_3}^{1/3} \left( \int_{\partial(-l_{x'},l_{x'})^2\times(0,l_{x_3})} |f - \frac{1}{l_{x_3}}\nu'\cdot(H'_{\mathrm{T}} - H'_{\mathrm{B}})|^2 dx' \right)^{4/3} \\ &\quad + l_{x_3}^{-1} \int_{(-l_{x'},l_{x'})^2} |H'_{\mathrm{T}} - H'_{\mathrm{B}}|^2 dx. \end{aligned}$$

Estimating the energy difference between the linear interpolation and the field instead of the full energy (or equivalently having constant 1 in the additional term in the second estimate) seems to have been a key improvement in the construction of comparison functions in [Con00] over [KM94]. It is crucial for being able to prove the required decay of the cumulated field $H'_{\mathrm{T}}$.

*Proof. 1. Rescaling.* We rescale to $l_{x_3} = 1$. This scales $x'$, $l_{x'}$, and $H$ by $l_{x_3}^{-2/3}$ and the energy by $l_{x_3}^{-5/3}$.

*2. Splitting the field boundary data.* We split $f$ into three parts: The first part is the $x_3$-average of $f$, i.e.

$$f_1(x') := \int_0^1 f(x', x_3) dx_3 = \nu' \cdot (H'_{\mathrm{T}}(x') - H'_{\mathrm{B}}(x')).$$

Next we split the $x_3$-oscillation $f - f_1$ into a high-frequency (w.r.t. $x'$) and a low-frequency part. We do this by decomposing $f$ into local averages and oscillation relative to that. We cover a boundary layer of size $s_{x',\mathrm{b}} = l_{x'} 2^{-N+1}$, leaving $N$ to be determined later, with squares

$$S_{\mathrm{b},i,j} = (i s_{x',\mathrm{b}}, (i+1) s_{x',\mathrm{b}}) \times (j s_{x',\mathrm{b}}, (j+1) s_{x',\mathrm{b}})$$

and define the local averages

$$f_2(x) := |\partial(-l_{x'}, l_{x'})^2 \cap \partial S_{\mathrm{b},i,j}|^{-1} \int_{\partial(-l_{x'},l_{x'})^2\cap\partial S_{\mathrm{b},i,j}} f(\xi', x_3) d\xi' \text{ for } x' \in \partial(-l_{x'}, l_{x'})^2 \cap \partial S_{\mathrm{b},i,j}.$$

We denote the oscillatory component by

$$f_3 := f - f_1 - f_2.$$

Note that this is an $L^2$-orthogonal projection (on each sector), in particular

$$\int_{\partial(-l_{x'},l_{x'})^2\cap\partial S_{\mathrm{b},i,j}} |f_2|^2 dx' + \int_{\partial(-l_{x'},l_{x'})^2\cap\partial S_{\mathrm{b},i,j}} |f_3|^2 dx' = \int_{\partial(-l_{x'},l_{x'})^2\cap\partial S_{\mathrm{b},i,j}} |f-f_1|^2 dx'.$$

We accomodate the high-frequency part $f_3$ by a divergence-free field, i.e. one not influencing the magnetization. In each slice we apply Lemma 16 on $S_{\mathrm{b},i,j}$ at the boundary. As the boundary flux we use the (zero-mean) oscillatory part $f_3$ on the one or two sides $\partial S_{\mathrm{b},i,j} \cap \partial(-l_{x'}, l_{x'})^2$ and zero on the other sides of $\partial S_{\mathrm{b},i,j}$. Note that $x_1^*$ of the lemma does not play a role here as the jump height would be zero. Lemma 16 yields a comparison field $h_0'$ with energy bounded as

$$\int_{S_{\mathrm{b},i,j}} |h_0'|^2 dx' \le C_{\mathrm{L16}} s_{x',\mathrm{b}} \int_{\partial S_{\mathrm{b},i,j}\cap\partial(-l_{x'},l_{x'})^2} |f_3|^2 dx'.$$

Adding up all boundary squares we have the bound

$$\int_{(-l_{x'},l_{x'})^2\setminus(-l_{x'}+s_{x',\mathrm{b}},l_{x'}-s_{x',\mathrm{b}})^2} |h_0'|^2 dx' \le C_{\mathrm{L16}} s_{x',\mathrm{b}} \int_{\partial(-l_{x'},l_{x'})^2} |f_3|^2 dx' \le C_{\mathrm{L16}} s_{x',\mathrm{b}} \int_{\partial(-l_{x'},l_{x'})^2} |f-f_1|^2 dx'.$$

By splitting out the highly oscillatory part we gain an $L^\infty$-estimate for the remaining low-frequency part $f_2$. Knowing $f_2$ is constant on $\partial S_{\mathrm{b},i,j} \cap \partial(-l_{x'}, l_{x'})^2$ we compute

$$\begin{aligned}
\sup_{\partial S_{\mathrm{b},i,j}\cap\partial(-l_{x'},l_{x'})^2} |f_2| &= |\partial S_{\mathrm{b},i,j} \cap \partial(-l_{x'}, l_{x'})^2|^{-1} \int_{\partial S_{\mathrm{b},i,j}\cap\partial(-l_{x'},l_{x'})^2} |f_2| dx' \\
&\le |\partial S_{\mathrm{b},i,j} \cap \partial(-l_{x'}, l_{x'})^2|^{-1/2} \left( \int_{\partial S_{\mathrm{b},i,j}\cap\partial(-l_{x'},l_{x'})^2} |f_2|^2 dx' \right)^{1/2} \\
&\le s_{x',\mathrm{b}}^{-1/2} \left( \int_{\partial S_{\mathrm{b},i,j}\cap\partial(-l_{x'},l_{x'})^2} |f-f_1|^2 dx' \right)^{1/2}, \qquad (83)
\end{aligned}$$

or if we prefer to just take the supremum over the full boundary

$$\sup_{x'\in\partial(-l_{x'},l_{x'})} |f_2| \le s_{x',\mathrm{b}}^{-1/2} \left( \int_{\partial(-l_{x'},l_{x'})^2} |f-f_1|^2 dx' \right)^{1/2}. \qquad (84)$$

We introduce the cumulated field flux across the boundary

$$F_2(x', x_3) := \int_0^{x_3} f_2(x', \xi_3) d\xi_3.$$

*3. An initial relaxed magnetization.* We start with the linear interpolation between $m_3$ and $H'$ at top and bottom. By construction

$$\begin{aligned}
m_3^{\mathrm{lin}} &:= x_3 m_3^{\mathrm{T}} + (1-x_3) m_3^{\mathrm{B}}, \\
h'_{\mathrm{lin}} &:= H'_{\mathrm{T}} - H'_{\mathrm{B}}, \\
H'_{\mathrm{lin}} &:= x_3 H'_{\mathrm{T}} + (1-x_3) H'_{\mathrm{B}}
\end{aligned}$$

satisfy

$$\begin{aligned} \partial_3 m_3^{\text{lin}} + \nabla' \cdot h'_{\text{lin}} &= 0 \text{ in } (-l_{x'}, l_{x'})^2 \times (0,1), \\ h'_{\text{lin}} \cdot \nu' &= f_1 = (H'_{\text{T}} - H'_{\text{B}}) \cdot \nu' \text{ on } \partial(-l_{x'}, l_{x'})^2 \times (0,1). \end{aligned}$$

Note that this $x_3$-constant field does provide a lower bound for the field energy.

We have to also accomodate the field flux $f_2$. To this end, consider again a square

$$S = S_{\text{b},i,j}$$

at the boundary. We want to change the magnetization within $S$ to obtain a valid relaxed magnetization $\tilde{m}_3$ (i.e. $|\tilde{m}_3| \leq 1$) and field such that

$$\partial_3 \tilde{m}_3 + \nabla' \cdot \tilde{h}' = 0$$

and

$$\nu' \cdot \tilde{h}' = \begin{cases} \nu' \cdot h'_{\text{lin}} + f_2 & \text{on } \partial S \cap \partial(-l_{x'}, l_{x'})^2, \\ \nu' \cdot h'_{\text{lin}} & \text{on } \partial S \setminus \partial(-l_{x'}, l_{x'})^2. \end{cases}$$

As we wish to simultaneously do this for all horizontal slices we need to change the total magnetization in $S$ to satisfy

$$\int_S \tilde{m}_3 dx' = \int_S m_3^{\text{lin}} dx' - \int_{\partial S \cap \partial(-l_{x'}, l_{x'})^2} F_2 dx'.$$

We want to change the linear magnetization proportionally to the distance (in the change direction $\pm 1$ in the image space) to the constraint $\tilde{m}_3 \in [-1, 1]$. We thus estimate the (double) "volume of each phase"

$$\begin{aligned} M^{\pm 1} &:= \left| \int_S (\pm 1 + m_3^{\text{lin}}) dx' \right| \\ &\geq |S| - \left| \int_S m_3^{\text{lin}} dx' \right| \\ &= |S| - \left| \int_S \nabla' \cdot H'_{\text{lin}} dx' \right| \\ &= |S| - \left| \int_{\partial S} \nu' \cdot H'_{\text{lin}} dx' \right| \\ &\geq |S| - |\partial S| \sup |H'_{\text{lin}}| \\ &= s_{x',\text{b}}^2 - 4 s_{x',\text{b}} \sup |H'_{\text{lin}}|, \end{aligned}$$

abusing the term volume of a phase by applying it to a relaxed magnetization. In order to be able to accomodate a sizable deviation from the average magnetization we want

$$s_{x',\text{b}} \geq \frac{16}{3} C_{H'} \geq \frac{16}{3} \sup |H'_{\text{lin}}| \tag{85}$$

to get

$$M^{\pm 1} \geq \frac{1}{4} s_{x',\text{b}}^2.$$

Note that

$$\partial_3 M^{\pm 1} = \pm \int_S \partial_3 m_3^{\text{lin}} dx' = \mp 1 \int_{\partial S} \nu' \cdot (H'_{\text{T}} - H'_{\text{B}}) dx',$$

in particular

$$|\partial_3 M^{\pm 1}| \leq 2 s_{x',\text{b}}^2. \tag{86}$$

To avoid overlap we prefer

$$s_{x',\mathrm{b}} \leq l_{x'}. \tag{87}$$

We can now treat $F_2$ by adding a fraction of $-m_3^{\mathrm{lin}} - \operatorname{sign} F_2$ to the magnetization, more precisely we define the additional relaxed magnetization as

$$m_3^{\mathrm{add}} := \begin{cases} \frac{|\partial S \cap \partial(-l_{x'},l_{x'})^2|}{M^{\operatorname{sign} F_2}}(-|F_2| m_3^{\mathrm{lin}} - F_2) & \text{in each } S = S_{\mathrm{b},i,j}, \\ 0 & \text{in } (-l_{x'} + s_{x',\mathrm{b}}, l_{x'} - s_{x',\mathrm{b}})^2 \end{cases}$$

and the relaxed magnetization

$$\tilde{m}_3 := m_3^{\mathrm{lin}} + m_3^{\mathrm{add}}. \tag{88}$$

Recall that $F_2$ is constant on each $\partial S_{\mathrm{b},i,j} \cap \partial(-l_{x'}, l_{x'})^2$ to see that there is no ambiguity in the definition.

We need to take care that $\tilde{m}_3 \in [-1,1]$. This is clear in $(-l_{x'} + s_{x',\mathrm{b}}, l_{x'} - s_{x',\mathrm{b}})^2$. In the remaining region we estimate by seeing that $\tilde{m}_3$ is a convex combination of $m_3^{\mathrm{lin}}$ and $-\operatorname{sign} F$. More precisely,

$$\tilde{m}_3 = \left(1 - \frac{|\partial S \cap \partial(-l_{x'}, l_{x'})^2|}{M^{\operatorname{sign} F_2}}|F_2|\right) m_3^{\mathrm{lin}} + \frac{|\partial S \cap \partial(-l_{x'}, l_{x'})^2|}{M^{\operatorname{sign} F_2}}|F_2|(-\operatorname{sign} F_2)$$

is a convex combination of values in $[-1,1]$ when

$$\frac{|\partial S \cap \partial(-l_{x'}, l_{x'})^2|}{M^{\operatorname{sign} F_2}}|F_2| \overset{(85)}{\leq} \frac{|\partial S \cap \partial(-l_{x'}, l_{x'})^2|}{\frac{1}{4}s_{x',\mathrm{b}}^2}|F_2| \leq \frac{2 s_{x',\mathrm{b}}}{\frac{1}{4}s_{x',\mathrm{b}}^2} \sup_{\partial S \cap \partial(-l_{x'}, l_{x'})^2} |F_2|$$

is bounded by 1. Thus we can ensure $\tilde{m}_3 \in [-1,1]$ by requiring

$$s_{x',\mathrm{b}} \geq 8 \sup_{(\partial S \cap \partial(-l_{x'}, l_{x'})^2) \times (0,1)} |F_2|. \tag{89}$$

We also specify a field $\tilde{h}'$ such that

$$\partial_3 \tilde{m}_3 + \nabla' \cdot \tilde{h}' = 0$$

by means of a correction $h'_{\mathrm{add}}$ to $h'_{\mathrm{lin}}$ such that

$$\begin{aligned} \partial_3 m_3^{\mathrm{add}} + \nabla' \cdot h'_{\mathrm{add}} &= 0 \text{ in each } S_{\mathrm{b},i,j}, \\ \nu' \cdot h'_{\mathrm{add}} &= f_2 \text{ on each } \partial S_{\mathrm{b},i,j} \cap \partial(-l_{x'}, l_{x'})^2, \\ \nu' \cdot h'_{\mathrm{add}} &= 0 \text{ on each } \partial S_{\mathrm{b},i,j} \setminus \partial(-l_{x'}, l_{x'})^2. \end{aligned}$$

Let us again fix $S = S_{\mathrm{b},i,j}$. We apply Lemma 16 on each boundary sector to obtain divergence-free fields $H'_{\mathrm{T},0}$ with normal component

$$f = \begin{cases} -\nu' \cdot H'_{\mathrm{T}} + |\partial S \cap \partial(-l_{x'}, l_{x'})^2|^{-1} \int_{\partial S} \nu' \cdot H'_{\mathrm{T}}\, dx' & \text{on } \partial S \cap \partial(-l_{x'}, l_{x'})^2, \\ -\nu' \cdot H'_{\mathrm{T}} & \text{on } \partial S \setminus \partial(-l_{x'}, l_{x'})^2 \end{cases}$$

and analogously $H'_{\mathrm{B},0}$ with $H'_{\mathrm{B}}$ in the boundary flux. From the lemma and (81) we know that

$$\sup |H'_{\mathrm{T},0}|, \sup |H'_{\mathrm{B},0}| \leq C C_{H'}. \tag{90}$$

As in the definition of $H'_{\mathrm{lin}}$ we write

$$H'_{\mathrm{lin},0} := x_3 H'_{\mathrm{T},0} + (1 - x_3) H'_{\mathrm{B},0}.$$

Let $H_1'$ be the unique linear field with

$$\begin{aligned}\nabla'\cdot H_1' &= 1 \text{ in } S,\\ \nu'\cdot H_1' &= \frac{|S|}{|\partial S\cap\partial(-l_{x'},l_{x'})^2|} \text{ on } \partial S\cap\partial(-l_{x'},l_{x'})^2,\\ \nu'\cdot H_1' &= 0 \text{ on } \partial S\setminus\partial(-l_{x'},l_{x'})^2,\end{aligned}$$

i.e.

$$H_1' = \begin{cases} (l_{x'}-s_{x',\mathrm{b}}+x_1)e_1 & \text{if } i=-2^{N-1}, j\notin\{-2^{N-1},2^{N-1}-1\},\\ (l_{x'}-s_{x',\mathrm{b}}-x_1)e_1 & \text{if } i=2^{N}-1, j\notin\{-2^{N-1},2^{N-1}-1\},\\ (l_{x'}-s_{x',\mathrm{b}}+x_2)e_2 & \text{if } j=-2^{N-1}, i\notin\{-2^{N-1},2^{N-1}-1\},\\ (l_{x'}-s_{x',\mathrm{b}}-x_2)e_2 & \text{if } j=2^{N-1}-1, i\notin\{-2^{N-1},2^{N-1}-1\},\\ \frac12(l_{x'}-s_{x',\mathrm{b}}+x_1)e_1+\frac12(l_{x'}-s_{x',\mathrm{b}}+x_2)e_2 & \text{if } i=-2^{N-1}, j=-2^{N-1},\\ \frac12(l_{x'}-s_{x',\mathrm{b}}+x_1)e_1+\frac12(l_{x'}-s_{x',\mathrm{b}}-x_2)e_2 & \text{if } i=-2^{N-1}, j=2^{N-1}-1,\\ \frac12(l_{x'}-s_{x',\mathrm{b}}-x_1)e_1+\frac12(l_{x'}-s_{x',\mathrm{b}}+x_2)e_2 & \text{if } i=2^{N-1}-1, j=-2^{N-1},\\ \frac12(l_{x'}-s_{x',\mathrm{b}}-x_1)e_1+\frac12(l_{x'}-s_{x',\mathrm{b}}-x_2)e_2 & \text{if } i=2^{N-1}-1, j=2^{N-1}-1.\end{cases}$$

Note that

$$\sup|H_1'|\le s_{x',\mathrm{b}} \tag{91}$$

and that the normal component of both $H'_{\mathrm{lin}}+H'_{\mathrm{lin},0}$ and $H_1'$ vanishes on $\partial S\setminus\partial(-l_{x'},l_{x'})^2$, so we can glue and extend by 0 without incurring a singular divergence term at these boundaries.

With these preparations we define

$$H'_{\mathrm{add}} := \begin{cases}\frac{|\partial S\cap\partial(-l_{x'},l_{x'})^2|}{M^{\operatorname{sign}F_2}}F_2\left(-\operatorname{sign}F_2(H'_{\mathrm{lin}}+H'_{\mathrm{lin},0})+H_1'\right) & \text{in } S=S_{\mathrm{b},i,j}\\ 0 & \text{in } (-l_{x'}+s_{x',\mathrm{b}},l_{x'}-s_{x',\mathrm{b}})^2,\end{cases}$$

satisfying $\nabla'\cdot H'_{\mathrm{add}}=-m_3^{\mathrm{add}}$ and compute the $x_3$-derivative $h'_{\mathrm{add}}:=\partial_3 H'_{\mathrm{add}}$ to get a field

$$h'_{\mathrm{add}} = \begin{cases}\frac{|\partial S\cap\partial(-l_{x'},l_{x'})^2|}{M^{\operatorname{sign}F_2}}f_2\left(-\operatorname{sign}F_2(H'_{\mathrm{lin}}+H'_{\mathrm{lin},0})+H_1'\right)\\ -\frac{|\partial S\cap\partial(-l_{x'},l_{x'})^2|}{M^{\operatorname{sign}F_2}}\frac{\partial_3 M^{\operatorname{sign}F_2}}{M^{\operatorname{sign}F_2}}F_2\left(-\operatorname{sign}F_2(H'_{\mathrm{lin}}+H'_{\mathrm{lin},0})+H_1'\right)\\ +\frac{|\partial S\cap\partial(-l_{x'},l_{x'})^2|}{M^{\operatorname{sign}F_2}}F_2\left(-\operatorname{sign}F_2((H'_{\mathrm{T}}+H'_{\mathrm{T},0})-(H'_{\mathrm{B}}+H'_{\mathrm{B},0}))\right) & \text{in } S=S_{\mathrm{b},i,j}\\ 0 & \text{in } (-l_{x'}+s_{x',\mathrm{b}},l_{x'}-s_{x',\mathrm{b}})^2,\end{cases}$$

to match $\partial_3 m_3^{\mathrm{add}}$. Observe that $m_3^{\mathrm{add}}$ vanishes where $\operatorname{sign}F_2$ changes. Thus we may assume that $\operatorname{sign}F_2$ is constant when computing the derivative because it indeed is for almost every $x_3$. The normal components of the last two summands exactly cancel at the boundary. Recall that $f_2$ is constant on each $\partial S_{\mathrm{b},i,j}\cap\partial(-l_{x'},l_{x'})^2$. The field

$$\tilde h' := h'_{\mathrm{lin}}+h'_{\mathrm{add}}$$

is compatible with the magnetization $\tilde m_3$ defined in (88). At the boundary $\partial(-l_{x'},l_{x'})^2\cap\partial S$ we have

$$\begin{aligned}\nu'\cdot\tilde h' &= \nu'\cdot h'_{\mathrm{lin}}+\frac{|\partial S\cap\partial(-l_{x'},l_{x'})^2|}{M^{\operatorname{sign}F_2}}f_2\frac{-\operatorname{sign}F_2\int_{\partial S}\nu'\cdot H'_{\mathrm{lin}}\,dx'+|S|}{|\partial S\cap\partial(-l_{x'},l_{x'})^2|}\\ &= \nu'\cdot h'_{\mathrm{lin}}+\frac{f_2}{M^{\operatorname{sign}F_2}}\int_S(-\operatorname{sign}F_2\ \nabla'\cdot H'_{\mathrm{lin}}+1)\,dx'\\ &= \nu'\cdot h'_{\mathrm{lin}}+\frac{f_2}{M^{\operatorname{sign}F_2}}\int_S(\operatorname{sign}F_2\ m_3^{\mathrm{lin}}+1)\,dx'\\ &= f_1+f_2,\end{aligned}$$

as desired. By (81), (85), (86), (90), and (91) we can estimate the additional field strength for the correction in the boundary layer as

$$\begin{aligned} &\sup_{S_{\mathrm{b},i,j}\subset(-l_{x'},l_{x'})^2\setminus(-l_{x'}+s_{x',\mathrm{b}},l_{x'}-s_{x',\mathrm{b}})^2} |h'_{\mathrm{add}}| \\ &\overset{(81),(86),(90),(91)}{\leq} \quad C(|f_2|+|F_2|)s_{x',\mathrm{b}}^{-1}(C_{H'}+s_{x',\mathrm{b}}) \\ &\overset{(85)}{\leq} \quad C(|f_2|+|F_2|). \end{aligned} \tag{92}$$

We emphasize that it would be premature to take the supremum over the full boundary layer at this point because we later want to integrate over the boundary layer and would lose a factor of essentially $l_{x'}$ if we took the supremum now.

*4. Subdivision of the domain and local averaging.* In preparation for defining a $\{-1,+1\}$-valued magnetization we locally modify $\tilde{m}_3$. We divide the cuboid $(-l_{x'},l_{x'})^2\times(0,1)$ in a way such that the pieces get smaller towards the boundary. It is natural to construct the magnetization on these pieces. Before we do that, however, we average the magnetization to be piecewise constant in $x'$ on each horizontal slice of such a cuboid sector while still being continuous in $x_3$ even across sector boundaries. We fully refine the structure as $x_3$ approaches either top or bottom boundary in order to get by without detailed knowledge about the magnetization $m_3^{\mathrm{T}}$ and $m_3^{\mathrm{B}}$ at the top and bottom boundaries. To make things precise, each layer is in an $x_3$-interval

$$I_{x_3,k} := (2^{-3(k+1)/2-1}, 2^{-3k/2-1}) \cup (1-2^{-3k/2-1}, 1-2^{-3(k+1)/2-1})$$

for $k\in\mathbb{N}_0$. We define the $x'$-lengthscale on each $I_{x_3,k}$

$$s_{x'} = 2^{-k_0+k+1}l_{x'}.$$

with $k_0$ chosen as the unique positive integer such that

$$\frac{1}{2} < 2^{-k_0+1}l_{x'} \leq 1.$$

We divide $(-l_{x'},l_{x'})$ into the subintervals

$$I^k_{x',j} = (js_{x'},(j+1)s_{x'})$$

and so split $(-l_{x'},l_{x'})^2$ into $2^{2(k+k_0)}$ subsquares

$$S'^k_{i,j} := I^k_{x',i}\times I^k_{x',j}.$$

Let

$$\alpha(x_3) := \begin{cases} \frac{x_3-2^{-3(k+1)/2-1}}{2^{-3k/2-1}-2^{-3(k+1)/2-1}} & \text{for } x_3\in I_{x_3,k}\cap[0,\frac{1}{2}], \\ \alpha(l_{x_3}-x_3) & \text{for } x_3\in I_{x_3,k}\cap(\frac{1}{2},1] \end{cases}$$

be the relative position of $x_3$ in $I_{x_3,k}$. We split each $I_{x_3,k}$ in two parts, one in which the local averaging of $\tilde{m}_3$ is refined and a second where the geometric refinement is done. This is done to simplify the somewhat technical construction and only costs a constant factor in our estimates. We thus introduce

$$\begin{aligned} \alpha_0(x) &:= \min\{2\alpha,1\}, \\ \alpha_1(x) &:= \max\{2\alpha-1,0\}. \end{aligned}$$

We define the envisioned average magnetization for $x$ such that $x_3 \in I_{x_3,k}$ and $x' \in {S'}^k_{i',j'} \subset {S'}^{k-1}_{i,j}$ as

$$\hat{m}_3(x',x_3) := (1-\alpha_0(x_3))|{S'}^k_{i',j'}|^{-1}\int_{{S'}^k_{i',j'}} \tilde{m}_3(\xi',x_3)d\xi' + \alpha_0(x_3)|{S'}^{k-1}_{i,j}|^{-1}\int_{{S'}^{k-1}_{i,j}} \tilde{m}_3(\xi',x_3)d\xi'.$$

Note that in particular $\int_{(-l_{x'},l_{x'})^2} \hat{m}_3(x',x_3)dx' = \int_{(-l_{x'},l_{x'})^2} \tilde{m}_3(x',x_3)dx'$ so the magnetization is compatible with the boundary conditions.

*5. Undoing the relaxation.* We now define a $\{+1,-1\}$-valued magnetization that has average $\hat{m}_3$. We treat two adjacent sectors ${S'}^k_{2i,j'} \cup {S'}^k_{2i+1,j'}$ at once. Let

$$m_3 := \begin{cases} +1 & \text{for } x_1 \in (s_{x'}2i, s_{x'}(2i+\min\{(1+\alpha_1)\frac{\hat{m}_3+1}{2},1\})) \\ & \qquad \cup (s_{x'}(2i+1), s_{x'}(2i+1+\max\{(1-\alpha_1)\frac{\hat{m}_3+1}{2},\hat{m}_3\})), \\ & \qquad x_2 \in I^k_{x',j'}, \\ -1 & \text{elsewhere on } {S'}^k_{2i,j'} \cup {S'}^k_{2i+1,j'}. \end{cases}$$

Note that $\hat{m}_3$ is constant on the union of the two sectors when $\alpha_1 \neq 0$. We see that the averages of $\hat{m}_3$ and $m_3$ agree, i.e.

$$\int_{{S'}^k_{2i,j'} \cup {S'}^k_{2i+1,j'}} \hat{m}_3 dx' = \int_{{S'}^k_{2i,j'} \cup {S'}^k_{2i+1,j'}} m_3 dx'$$

and so on the larger sectors ${S'}^{k-1}_{i,j}$ we have

$$\int_{{S'}^{k-1}_{i,j}} \hat{m}_3 dx' = \int_{{S'}^{k-1}_{i,j}} m_3 dx' = \int_{{S'}^{k-1}_{i,j}} \tilde{m}_3 dx'.$$

We compute $\partial_3 m_3$ in the sense of distributions and see that it can be represented as

$$\begin{aligned} \partial_3 m_3 &= 2\partial_3 x_1^{(2i,j,k)}(x_3) \cdot \mathcal{H}^1 \llcorner \{x_1 = x_1^{(2i,j,k)}(x_3)\} \\ &\quad + 2\partial_3 x_1^{(2i+1,j,k)}(x_3) \cdot \mathcal{H}^1 \llcorner \{x_1 = x_1^{(2i+1,j,k)}(x_3)\} \end{aligned}$$

when we define

$$\begin{aligned} x_1^{(2i,j,k)}(x_3) &:= s_{x'}(2i+\min\{(1+\alpha_1)\frac{\hat{m}_3+1}{2},1\}), \\ x_1^{(2i+1,j,k)}(x_3) &:= s_{x'}(2i+1+\max\{(1-\alpha_1)\frac{\hat{m}_3+1}{2},\hat{m}_3\}). \end{aligned}$$

*6. Comparison field.* As the construction of $m_3$ from $\tilde{m}_3$ preserves slicewise averages in $S^{k-1}_{i,j} \times I_{x_3,k}$ our strategy is to construct the comparison field using a local modification $\tilde{h}'$ to deal with the change in $\hat{m}_3$. Let us write

$$(f)_{i,j,k} := |{S'}^k_{i,j}|^{-1}\int_{{S'}^k_{i,j}} f d\xi'.$$

For the following calculations we fix $i$, $j$ and let $i'$ and $j'$ vary in $\{2i,2i+1\}$ and $\{2j,2j+1\}$, respectively. We compute the first density $2\partial_3 x_1^{(2i,j',k)}$ if ${S'}^k_{2i,j'}$ has not filled up with $+1$ magnetization, i.e.

$x_1^{(2i,j',k)} < (2i+1)s_{x'}$ or $\alpha_0 < 1$,

$$\begin{aligned}
2\partial_3 x_1^{(2i,j',k)} &= s_{x'}\partial_3\alpha_1(\hat m_3+1) + s_{x'}(1+\alpha_1)\partial_3\hat m_3 \\
&= s_{x'}\partial_3\alpha_1(\hat m_3+1) + s_{x'}(1+\alpha_1)\partial_3\left((\tilde m_3)_{(2i,j',k)} + \alpha_0((\tilde m_3)_{i,j,k-1} - (\tilde m_3)_{2i,j',k})\right) \\
&= s_{x'}\partial_3\alpha_1(\hat m_3+1) + s_{x'}(1+\alpha_1)\partial_3\alpha_0((\tilde m_3)_{i,j,k-1} - (\tilde m_3)_{2i,j',k}) \\
&\quad + s_{x'}(1+\alpha_1)\left((1-\alpha_0)(\partial_3\tilde m_3)_{(2i,j',k)} + \alpha_0((\partial_3\tilde m_3)_{i,j,k-1})\right) \\
&= s_{x'}\partial_3\alpha_1(\hat m_3+1) + s_{x'}\partial_3\alpha_0((\tilde m_3)_{i,j,k-1} - (\tilde m_3)_{2i,j',k}) \\
&\quad + s_{x'}\left((1-\alpha_0)(\partial_3\tilde m_3)_{(2i,j',k)} + (1+\alpha_1)\alpha_0((\partial_3\tilde m_3)_{i,j,k-1})\right),
\end{aligned}$$

where we have used that at any point either $\alpha_1 = \partial_3\alpha_1 = 0$ or $\alpha_0 = 1$ and $\partial_3\alpha_0 = 0$.

Similarly we compute the second density in the case that $x_1^{(2i,j',k)} < (2i+1)s_{x'}$ or $\alpha_0 < 1$, i.e. when the maximum in the expression for $x_1^{(2i+1,j',k)}$ is the first argument,

$$\begin{aligned}
2\partial_3 x_1^{(2i+1,j',k)} &= -s_{x'}\partial_3\alpha_1(\hat m_3+1) + s_{x'}(1-\alpha_1)\partial_3\hat m_3 \\
&= -s_{x'}\partial_3\alpha_1(\hat m_3+1) + s_{x'}(1-\alpha_1)\partial_3\alpha_0((\tilde m_3)_{i,j,k-1} - (\tilde m_3)_{2i+1,j',k}) \\
&\quad + s_{x'}(1-\alpha_1)\left((1-\alpha_0)(\partial_3\tilde m_3)_{(2i+1,j',k)} + \alpha_0((\partial_3\tilde m_3)_{i,j,k-1})\right) \\
&= -s_{x'}\partial_3\alpha_1(\hat m_3+1) + s_{x'}\partial_3\alpha_0((\tilde m_3)_{i,j,k-1} - (\tilde m_3)_{2i+1,j',k}) \\
&\quad + s_{x'}\left((1-\alpha_0)(\partial_3\tilde m_3)_{(2i+1,j',k)} + (1-\alpha_1)\alpha_0((\partial_3\tilde m_3)_{i,j,k-1})\right).
\end{aligned}$$

In the other case, $x_1^{(2i,j',k)} = (2i+1)s_{x'}$ and $\alpha_0 = 1$, the density is instead

$$\begin{aligned}
2\partial_3 x_1^{(2i+1,j',k)} &= 2s_{x'}\partial_3\hat m_3 \\
&= 2s_{x'}(\partial_3\tilde m_3)_{i,j,k-1}.
\end{aligned}$$

We construct the comparison field as a sum of fields reflecting this decomposition of $\partial_3 x_1^{(2i,j',k)}$ and $\partial_3 x_1^{(2i+1,j',k)}$. Denoting by $e_i$ the standard unit vectors we see that

$$h'_{\mathrm{a}} = \begin{cases} -s_{x'}\partial_3\alpha_1(\hat m_3+1)e_1 & \text{for } x_1 \in (x_1^{(2i,j,k)}, x_1^{(2i+1,j,k)}), x_1^{(2i,j,k)} < s_{x'}(2i+1), \\ 0 & \text{otherwise} \end{cases}$$

compensates the first summand (featuring $\partial_3\alpha_1$) if it occurs. The strength of this field is bounded by

$$|h'_{\mathrm{a}}| \le 4s_{x'}s_{x_3}^{-1}.$$

For the term involving $\partial_3\alpha_0$ we consider the four squares ${S'}^k_{2i,2j}, {S'}^k_{2i+1,2j}, {S'}^k_{2i,2j+1}, {S'}^k_{2i+1,2j+1}$ comprising ${S'}^{k-1}_{i,j}$. We note that $\alpha_1 = 0$ if $\partial\alpha_0 \ne 0$. Let

$$h'_{\mathrm{b},1,i,j'}(x) = \begin{cases} -s_{x'}\partial_3\alpha_0((\tilde m_3)_{i,j,k-1} - (\tilde m_3)_{2i,j',k})e_1 & \text{in } {S'}^k_{2i,j'} \cap \{x_1 \ge x_1^{(2i,j',k)}\}, \\ s_{x'}\partial_3\alpha_0((\tilde m_3)_{i,j,k-1} - (\tilde m_3)_{2i+1,j',k})e_1 & \text{in } {S'}^k_{2i+1,j'} \cap \{x_1 \le x_1^{(2i+1,j',k)}\}, \\ 0 & \text{elsewhere in } {S'}^k_{2i,j'} \cup {S'}^k_{2i+1,j'} \end{cases}$$

and

$$h'_{\mathrm{b},2,i,j}(x) = \begin{cases} -s_{x'}\partial_3\alpha_0(2(\tilde m_3)_{i,j,k-1} - (\tilde m_3)_{2i,2j,k} - (\tilde m_3)_{2i+1,2j,k})(e_1+e_2) \\ \qquad \text{in } {S'}^k_{2i+1,2j} \cap \{x_2 - 2js_{x'} \ge x_1 - (2i+1)s_{x'}\}, \\ -s_{x'}\partial_3\alpha_0(2(\tilde m_3)_{i,j,k-1} - (\tilde m_3)_{2i,2j+1,k} - (\tilde m_3)_{2i+1,2j+1,k})(e_1-e_2) \\ \qquad \text{in } {S'}^k_{2i+1,2j+1} \cap \{x_2 - (2j+1)s_{x'} \le (2i+2)s_{x'} - x_1\}, \\ 0 \qquad \text{elsewhere.} \end{cases}$$

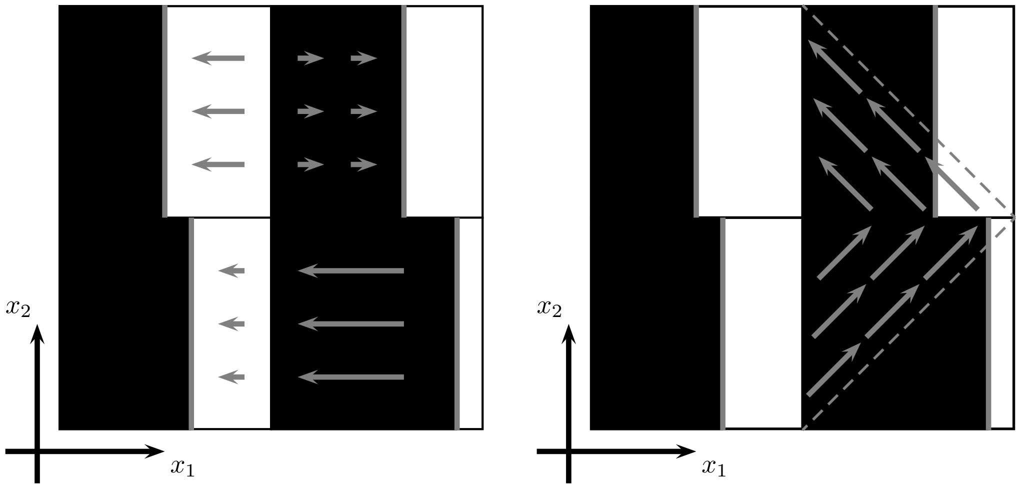

Figure 12: Construction of $h'_{\mathrm{b},1}$ (left) and $h'_{\mathrm{b},2}$ (right), black regions have magnetization $m_3 = +1$, (arrow lengths are not drawn to scale)

Then

$$h'_{\mathrm{b}} = h'_{\mathrm{b},1,i,2j} + h'_{\mathrm{b},1,i,2j+1} + h'_{\mathrm{b},2,i,j}$$

(see also Figure 12) matches the terms of the derivative of $m_3$ involving $\partial_3\alpha_0$ and we can bound the field strength as

$$|h'_{\mathrm{b}}| \le 12 s_{x'} |\partial_3\alpha_0| \le 24 s_{x'} s_{x_3}^{-1}$$

without trying to get a good constant.

Finally, we want to correct $\tilde{h}$ to match the non-relaxed magnetization, i.e. find a field corresponding to the terms involving $\partial_3\tilde{m}_3$ in the density of the measure representing $\partial_3 m_3$. As the reader might suspect after our construction of $\tilde{h}$, we adjust $h'_{\mathrm{lin}}$ and $\tilde{h}' - h'_{\mathrm{lin}}$ separately.

We start with $h'_{\mathrm{lin}}$. We apply Lemma 15 on ${S'}_{(i',j',k)}$ with

$$g = \nabla' \cdot h'_{\mathrm{lin}} = -\partial_3 m^{\mathrm{lin}} = -(m_3^{\mathrm{T}} - m_3^{\mathrm{B}})$$

and $M = \{x_1^{(i',j',k)}\} \times I^k_{x',j'}$ to find $h'_{\mathrm{c},1,i',j'} := h'_{\mathrm{lin}} + \nabla' w$. With

$$\int_{{S'}_{(i',j',k)}} |g|^2 dx' \le 2|{S'}_{(i',j',k)}| = 2 s_{x'}^2$$

the bound of Lemma 15 is

$$\int_{{S'}_{(i',j',k)}} |h'_{\mathrm{lin}} - h'_{\mathrm{c},1,i',j'}|^2 \le C s_{x'}^4$$

and we achieve

$$\nabla' \cdot h'_{\mathrm{c},1,i',j'} = -s_{x'} (\partial_3\tilde{m}_3)_{i',j',k} \cdot \mathcal{H}^1 \llcorner M \text{ on } {S'}^k_{i',j'}.$$

We apply Lemma 15 a second time, now on ${S'}^{k-1}_{i,j}$ with $g = \nabla \cdot h'_{\mathrm{lin}}$ and the same $M \subset {S'}^k_{i',j'}$ to obtain $h'_{\mathrm{c},2,i',j'} := h'_{\mathrm{lin}} + \nabla w$ with the same estimate (but the constant being 16 times larger) and

$$\nabla' \cdot h'_{\mathrm{c},2,i',j'} = -4 s_{x'} (\partial_3\tilde{m}_3)_{i,j,k-1} \cdot \mathcal{H}^1 \llcorner M \text{ on } {S'}^{k-1}_{i,j}.$$

In the boundary sectors we also need to take care of $h'_{\text{add}}$. We proceed in a slighly different way keeping an eye on the maximal field strength instead of the energy per sector. We apply Lemma 16 on $S'^k_{i',j'}$ with $f = \nu' \cdot h'_{\text{add}}$, $x_1^* = x_1^{(i',j',k)}$ and $M = I^k_{x',j'}$ to obtain $h'_{\text{d},1,i',j'}$. Similarly, we apply Lemma 16 on $S'^{k-1}_{i,j}$ with the same $f$, $x_1^*$, and $M$ to obtain $h'_{\text{d},2,i',j'}$. We have the estimate pair

$$\sup_{S'^k_{i',j'}} |h'_{\text{d},1,i',j'}| \leq C_{\text{L16}} \sup_{S'^k_{i',j'}} |h'_{\text{add}}|$$

and

$$\sup_{S'^{k-1}_{i',j'}} |h'_{\text{d},2,i',j'}| \leq 2C_{\text{L16}} \sup_{S'^{k-1}_{i,j}} |h'_{\text{add}}|.$$

For $S = S'^k_{i',j'}$ or $S = S'^{k-1}_{i,j}$ we have

$$\int_{\partial S} \nu' \cdot \tilde{h}' dx' = \int_S \nabla' \cdot \tilde{h}' dx' = -\int_S \partial_3 \tilde{m}_3 dx'$$

and so

$$\begin{aligned}
\nabla' \cdot h'_{\text{d},1,i',j'} &= s_{x'}^{-1} \int_{\partial S'^k_{i',j'}} \nu' \cdot h'_{\text{d},1,i',j'} dx' \cdot \mathcal{H}^1 \llcorner \{x_1^{(i',j',k)}\} \times I^k_{x',j'} \\
&= -s_{x'}^{-1} \int_{S'^k_{i',j'}} \partial_3 m_3^{\text{add}} dx' \cdot \mathcal{H}^1 \llcorner \{x_1^{(i',j',k)}\} \times I^k_{x',j'} \\
&= -s_{x'} (\partial_3 m_3^{\text{add}})_{i',j',k} \cdot \mathcal{H}^1 \llcorner \{x_1^{(i',j',k)}\} \times I^k_{x',j'}
\end{aligned}$$

and similarly

$$\nabla' \cdot h'_{\text{d},2,i',j'} = -4 s_{x'} (\partial_3 \tilde{m}_3^{\text{add}})_{i',j',k} \cdot \mathcal{H}^1 \llcorner \{x_1^{(i',j',k)}\} \times I^k_{x',j'}.$$

Consistently with $h'_{\text{add}} \equiv 0$ in $(-l_{x'} + s_{x',\text{b}}, l_{x'} - s_{x',\text{b}})^2$ we define all $h'_{\text{d},\dots}$ to vanish there. Recall that we require that $2s_{x'}$ divides $s_{x',\text{b}}$, so any $S'^{k-1}_{i,j}$ is contained either in the boundary layer or its complement.

We see that the convex combination

$$h'_{\text{c}} = \begin{cases} (1-\alpha_0) h'_{\text{c},2,i',j'} \\ \qquad +(1+\alpha_1)\alpha_0 \frac{1}{4}(h'_{\text{c},2,2i,2j} + h'_{\text{c},2,2i,2j+1}) \\ \qquad +(1-\alpha_1)\alpha_0 \frac{1}{4}(h'_{\text{c},2,2i,2j} + h'_{\text{c},2,2i,2j+1}) & \text{on } S'^k_{i',j'} \text{ if } x_1^{2i,j',k} < (2i+1)s_{x'} \text{ or } \alpha_0 < 1, \\ \frac{1}{2}(h'_{\text{c},2,2i+1,2j} + h'_{\text{c},2,2i+1,2j+1}) & \text{on } S'^{k-1}_{i,j} \text{ if } x_1^{2i,j',k} = (2i+1)s_{x'} \text{ and } \alpha_0 = 1 \end{cases}$$

and $h_{\text{d}}$ defined as a convex combination with the very same coefficients are fields compensating the terms involving $\partial_3 m_3^{\text{lin}}$ and $\partial_3 m_3^{\text{add}}$, respectively, so their sum compensates $\partial_3 \tilde{m}_3$ in the derivative of $\partial_3 m_3$ on $S'^{k-1}_{i,j}$. With $\nu' \cdot (h'_{\text{c}} + h'_{\text{d}}) = \nu' \cdot \tilde{h}'$ we can glue the field for all sectors and obtain a field matching our boundary conditions.

We have treated these two components separately to be able to bound their energy contribution in different ways. The convex-combination of the bounds from Lemma 15 gives

$$\int_{S'^k_{i',j'}} |h'_{\text{c}} - h'_{\text{lin}}| dx' \leq C s_{x'}^4.$$

Lemma 16 lets $h'_{\text{d}}$ inherit the $L^\infty$-bound (92) from $h'_{\text{add}}$, i.e.

$$\sup_{S'^{k-1}_{i,j}} |h'_{\text{d}}| \leq 2C_{\text{L16}} \sup_{S'^{k-1}_{i,j}} |h'_{\text{add}}| \overset{(92)}{\leq} C(|f_2| + |F_2|)$$

for each ${S'}^{k-1}_{i,j} \subseteq S_{\mathrm{b},i_b,j_b} \subset (-l_{x'},l_{x'})^2 \setminus (-l_{x'}+s_{x',\mathrm{b}}, l_{x'}-s_{x',\mathrm{b}})^2$.

Summing up, we have constructed in each slice

$$h'_{\mathrm{abcd}} := h'_{\mathrm{a}} + h'_{\mathrm{b}} + h'_{\mathrm{c}} + h'_{\mathrm{d}}$$

satisfying

$$\begin{aligned} \partial_3 m_3 + \nabla' \cdot h'_{\mathrm{abcd}} &= 0 \text{ in } (-l_{x'},l_{x'})^2, \\ \nu' \cdot h'_{\mathrm{abcd}} = \nu' \cdot \tilde{h}' &= f_1 + f_2 \text{ on } \partial(-l_{x'},l_{x'})^2. \end{aligned}$$

Adding the divergence free $h'_0$ in the boundary squares, i.e. letting

$$h' := \begin{cases} h'_{\mathrm{abcd}} & \text{in } (-l_{x'}+s_{x',\mathrm{b}}, l_{x'}-s_{x',\mathrm{b}})^2, \\ h'_{\mathrm{abcd}} + h'_0 & \text{in } (-l_{x'},l_{x'})^2 \setminus (-l_{x'}+s_{x',\mathrm{b}}, l_{x'}-s_{x',\mathrm{b}})^2 \end{cases}$$

we have the desired comparison field $h'$ with

$$\begin{aligned} \partial_3 m_3 + \nabla' \cdot h' &= 0 \text{ in } (-l_{x'},l_{x'})^2, \\ \nu' \cdot h' &= f \text{ on } \partial(-l_{x'},l_{x'})^2. \end{aligned}$$

*7. Energy bound and boundary layer size.* In the interior we estimate $h' - h'_{\mathrm{lin}}$ because we want to exploit the orthogonality with $h'_{\mathrm{lin}}$ once we integrate in $x_3$. We thus bound the field energy as

$$\begin{aligned} \int_{(-l_{x'},l_{x'})^2} |h' - h'_{\mathrm{lin}}|^2 dx' &\lesssim \int_{(-l_{x'},l_{x'})^2} |h'_{\mathrm{a}} + h'_{\mathrm{b}}|^2 + |h'_{\mathrm{c}} - h'_{\mathrm{lin}}|^2 dx' \\ &\quad + \int_{(-l_{x'},l_{x'})^2 \setminus (-l_{x'}+s_{x',\mathrm{b}}, l_{x'}-s_{x',\mathrm{b}})^2} |h'_{\mathrm{d}}|^2 + |h'_0|^2 dx' \\ &\lesssim s_{x'}^2 s_{x_3}^{-2} l_{x'}^2 + s_{x'}^2 l_{x'}^2 + \sum_{S_{\mathrm{b},i,j}} s_{x',\mathrm{b}} |F_2|^2 \\ &\quad + s_{x',\mathrm{b}} \int_{\partial(-l_{x'},l_{x'})^2} |f - f_1|^2 dx' \\ &\lesssim s_{x'}^2 s_{x_3}^{-2} l_{x'}^2 + s_{x',\mathrm{b}} \int_{\partial(-l_{x'},l_{x'})^2} |f - f_1|^2 dx' + \sum_{S_{\mathrm{b},i,j}} s_{x',\mathrm{b}} |F_2|^2. \end{aligned}$$

Recall that $F_2$ is constant on $\partial S_{\mathrm{b},i,j} \cap \partial\partial(-l_{x'},l_{x'})^2$ for each $S_{\mathrm{b},i,j}$. The interfaces are the sector boundaries and (at most) one line through each sector. We thus have

$$2 \int_{[-l_{x'},l_{x'})^2} |\nabla' m_3| dx' \lesssim s_{x'}^{-1} l_{x'}^2.$$

Adding interfacial and field energy, plugging in the $x_3$-dependent lengthscales $s_{x'}$ and $s_{x_3}$, and integrating over $x_3$ we see that

$$\begin{aligned} &2 \int_{[-l_{x'},l_{x'})^2 \times (0,1)} |\nabla' m_3| dx + \int_{(-l_{x'},l_{x'})^2 \times (0,1)} |h' - h'_{\mathrm{lin}}|^2 dx \\ &\lesssim l_{x'}^2 \int_0^1 (s_{x'}^{-1}(x_3) + (s_{x'}(x_3))^2 s_{x_3}(x_3)^{-2}) dx_3 + s_{x',\mathrm{b}} \int_{\partial(-l_{x'},l_{x'})^2 \times (0,1)} |f - f_1|^2 dx' + \sum_{S_{\mathrm{b},i,j}} s_{x',\mathrm{b}} |F_2|^2 \\ &\lesssim l_{x'}^2 + s_{x',\mathrm{b}} \int_{\partial(-l_{x'},l_{x'})^2 \times (0,1)} |f - f_1|^2 dx'. \end{aligned}$$

Before we can complete the estimate, we have to collect our assumptions on $C_{H'}$, $l_{x'}$, and $s_{x',\mathrm{b}}$. We want to choose

$$s_{x',\mathrm{b}} \geq 4 \left( \int_{\partial(-l_{x'},l_{x'})^2\times(0,1)} |f - f_1|^2 dx \right)^{1/3}$$

because then by (84)

$$8 \sup_{\partial(-l_{x'},l_{x'})^2\times(0,1)} |F_2| \overset{(84)}{\leq} 4 \left( \int_{\partial(-l_{x'},l_{x'})^2\times(0,1)} |f - f_1|^2 dx \right)^{1/3} \leq s_{x',\mathrm{b}},$$

which we required in (89). Combined with the restriction of (85) and our desire that $2s_{x'}$ divides $s_{x',\mathrm{b}}$ we choose $s_{x',\mathrm{b}} = 2^{-N+1} l_{x'}$ with $N$ the unique integer such that

$$\frac{1}{2} s_{x',\mathrm{b}} < \max \left\{ 4 \left( \int_{\partial(-l_{x'},l_{x'})^2\times(0,1)} |f - f_1|^2 dx \right)^{1/3}, \frac{16}{3} C_{H'}, 2 \right\} \leq s_{x',\mathrm{b}}. \tag{93}$$

To be able to drop the last restriction we ask that

$$C_{H'} \geq 1.$$

Per (87) we want the boundary layer to not occupy too much of the domain, so we impose

$$l_{x'} \geq \frac{16}{3} C_{H'}$$

and, because we want to interpret the required relation as a bound on the boundary energy in terms of $l_{x'}$,

$$\int_{\partial(-l_{x'},l_{x'})^2\times(0,1)} |f - f_1|^2 dx \leq 2^{-9} l_{x'}^3.$$

Undoing the rescaling, we have the first energy estimate of the proposition.

It remains to consider the full energy instead of $h' - h'_{\mathrm{lin}}$ for the second formulation. Similar to Lemma 14 we consider our constructed comparison field as a comparison field for the minimization of the field energy with given magnetization $m_3$. For $H'_{\mathrm{T}} - H'_{\mathrm{B}}$ to be curl-free is equivalent to it having minimal $L^2$-norm for given magnetization difference $m_3^{\mathrm{T}} - m_3^{\mathrm{B}}$ and normal component on $\partial(-l_{x'}, l_{x'})^2$. By Lemma 14 the optimal field $h'_{\mathrm{opt}}$ for given $m_3$ has an $x_3$-average that is orthogonal to the $x_3$-oscillation and thus solves the same minimization problem. By the uniqueness of the minimizer the average has to coincide with $\frac{1}{l_{x_3}}(H'_{\mathrm{T}} - H'_{\mathrm{B}}) = h'_{\mathrm{lin}}$ and optimality of the oscillation implies

$$\begin{aligned}
\int_{(-l_{x'},l_{x'})^2\times(0,l_{x_3})} |h'_{\mathrm{opt}}|^2 dx &= \int_{(-l_{x'},l_{x'})^2\times(0,l_{x_3})} |h'_{\mathrm{opt}} - h'_{\mathrm{lin}}|^2 dx + \int_{(-l_{x'},l_{x'})^2\times(0,l_{x_3})} |h'_{\mathrm{lin}}|^2 dx \\
&\leq \int_{(-l_{x'},l_{x'})^2\times(0,l_{x_3})} |h' - h'_{\mathrm{lin}}|^2 dx + \int_{(-l_{x'},l_{x'})^2\times(0,l_{x_3})} |h'_{\mathrm{lin}}|^2 dx.
\end{aligned}$$

Plugging this into the first estimate we obtain the second. □

The following two lemmas are used in the construction of the field for the magnetization above. The first is a dual estimate to the Poincaré inequality in the form of Lemma 20 combined with the usual $L^2$–estimate for the solution of the Poisson equation.

**Lemma 15.** *Given $g \in L^2((0,l)^2)$ and $M = \{x_1^*\} \times (a, a+b) \subset (0,l)^2$ let $w$ be the solution to*

$$\begin{aligned} -\Delta' w &= g - \left( |M|^{-1} \int_{(0,l)^2} g dx' \right) \cdot \mathcal{H}^1 \llcorner M \text{ distributionally in } (0,l)^2, \\ \partial_{\nu'} w &= 0 \text{ on } \partial(0,l)^2, \\ \int_{(0,l)^2} w dx' &= 0. \end{aligned}$$

*Then with the universal constant $C = C(2,2)$ of Lemma 20*

$$\int_{(0,l)^2} |\nabla' w|^2 dx' \le C l^3 |M|^{-1} \int_{(0,l)^2} |g|^2 dx'.$$

*Proof.* We denote the density we want to put on $M$ by

$$\bar{g} := |M|^{-1} \int_{(0,l)^2} g dx'.$$

Without loss of generality we assume $x_1^* < l$ and let, for small $\varepsilon$,

$$\chi := \chi_{(x_1^*, x_1^* + \varepsilon \bar{g}) \times (a, a+b)}.$$

For $x_1^* = l$ we could use $\chi = \chi_{(x_1^* - \varepsilon \bar{g}, x_1^*) \times (a, a+b)}$ instead. Let $w_\varepsilon$ be the solution to

$$\begin{aligned} -\Delta' w_\varepsilon &= g - \varepsilon^{-1} \chi \text{ in } (0,l)^2, \\ \partial_{\nu'} w_\varepsilon &= 0 \text{ on } \partial(0,l)^2, \\ \int_{(0,l)^2} w_\varepsilon dx' &= 0. \end{aligned}$$

We estimate the $L^2$-norm of the gradient by testing with

$$\varphi = \|\nabla' w_\varepsilon\|_{L^2}^{-1} \left( w_\varepsilon - \left( \int \chi dx' \right)^{-1} \int w_\varepsilon \chi dx' \right).$$

Using the Poincaré inequality from Lemma 20 we see that

$$\begin{aligned}
\left(\int_{(0,l)^2} |\nabla' w_\varepsilon|^2 dx'\right)^{1/2} &= \int_{(0,l)^2} \nabla' w_\varepsilon \nabla' \varphi dx' \\
&= -\int_{(0,l)^2} \Delta' w_\varepsilon \varphi dx' \\
&= \int_{(0,l)^2} g\varphi dx' - \int_{(0,l)^2} \varepsilon^{-1}\chi\varphi dx' \\
&= \int_{(0,l)^2} g\varphi dx' \\
&\leq \left(\int_{(0,l)^2} |g|^2 dx'\right)^{1/2} \left(\int_{(0,l)^2} |\varphi|^2 dx'\right)^{1/2} \\
&\overset{\text{Lemma 20}}{\leq} C^{1/2} l \frac{l^{1/2}}{|M|^{1/2}} \left(\int_{(0,l)^2} |g|^2 dx'\right)^{1/2} \left(\int_{(0,l)^2} |\nabla'\varphi|^2 dx'\right)^{1/2} \\
&= C^{1/2} l \frac{l^{1/2}}{|M|^{1/2}} \left(\int_{(0,l)^2} |g|^2 dx'\right)^{1/2}.
\end{aligned}$$

For $\varepsilon \to 0$ the equation converges to

$$-\Delta' w = g - \bar{g}\mathcal{H}^1 \llcorner M \text{ in } (0,l)^2$$

and the solutions to the linear equation converge $w_\varepsilon \overset{w}{\rightharpoonup} w$ weakly in $H^1$, so we have

$$\int_{(0,l)^2} |\nabla' w|^2 dx' \leq C l^2 \frac{l}{|M|} \int_{(0,l)^2} |g|^2 dx',$$

the desired estimate. □

**Lemma 16.** *Given a square $S = (0,l)^2$, a function $f$ on $\partial S$, $x_1^* \in (0,l)$, and $M \subset (0,l)$ a finite union of intervals of measure $\mathcal{H}^1(M) = \alpha l$, there exists $h'$ such that*

$$\begin{aligned}
\nabla' \cdot h' &= (\alpha l)^{-1} \int_{\partial S} f dx' \cdot \mathcal{H}^1 \llcorner \{x_1^*\} \times M, \\
\nu' \cdot h' &= f \, on \, \partial S
\end{aligned}$$

*satisfying*

$$\sup_S |h'| \leq C_{L16}\, \alpha^{-1} \sup_{\partial S} |f|$$

*and if $\alpha = 1$*

$$\int_S |h'|^2 dx' \leq C_{L16}\, l \int_{\partial S} |f|^2 dx'.$$

*The constant $C_{L16}$ is universal (e.g. the first estimate would work with $C_{L16} = 16$).*

The condition that $M$ is a finite union of intervals is motivated by the application in the proof of Lemma 15 and we would expect the assertion of the lemma to hold under more general conditions.

*Proof.* Without loss of generality $x_1^* \leq \frac{1}{2}l$. Define

$$h_\mathrm{a}'(x') = (-f(0,x_2) - f(x_1 - x_2, 0) - f(x_1 + x_2 - l, l))e_1 + (f(x_1 - x_2, 0) - f(x_1 + x_2 - l, l))e_2$$

with the convention that $f$ vanishes outside $\partial S$ and

$$\tilde f(x_2) = f(l, x_2) - \nu' \cdot h_\mathrm{a}'.$$

Let $h_\mathrm{b}$ be the rotated gradient of

$$\psi(x') = \big(1 - \frac{x_1 - x_1^*}{l - x_1^*}\big) \int_0^l \tilde f(\xi_2) d\xi_2 (\alpha l)^{-1} \int_0^{x_2} \chi_M(\xi_2) d\xi_2 + \frac{x_1 - x_1^*}{l - x_1^*} \int_0^{x_2} \tilde f(\xi_2) d\xi_2$$

on $\{x' | x_1 \in (x_1^*, l)\}$ extended to $(0,l)^2$ by zero, i.e.

$$h_\mathrm{b}'(x') = \begin{cases} \Big(\big(1 - \frac{x_1 - x_1^*}{l - x_1^*}\big) \int_0^l \tilde f(\xi_2) d\xi_2 (\alpha l)^{-1} \chi_M(x_2) + \frac{x_1 - x_1^*}{l - x_1^*} \tilde f(x_2)\Big) e_1 & \\ + \frac{1}{l - x_1^*} \Big(\int_0^{x_2} \tilde f(\xi_2) d\xi_2 - \int_0^l \tilde f(\xi_2) d\xi_2 (\alpha l)^{-1} \int_0^{x_2} \chi_M(\xi_2) d\xi_2\Big) e_2 & \text{if } x_1 > x_1^*, \\ 0 & \text{otherwise.} \end{cases}$$

We see that $h' = h_\mathrm{a}' + h_\mathrm{b}'$ satisfies the right boundary conditions and with the estimates

$$|h_\mathrm{a}'| \lesssim \sup |f|$$

and (using $\alpha \leq 1$)

$$|h_\mathrm{b}'| \lesssim \sup |\tilde f| \lesssim |f|$$

we have the desired $L^\infty$-estimate. Similarly

$$\int_S |h'|^2 dx' \lesssim \int_S |h_b'|^2 + |h_b'|^2 dx' \lesssim l \int_{\partial S} |f|^2 dx',$$

proving our claim. □

The following lemma enhances the sketch of the ODE argument of the previous subsection to a rigorous proof. We have two options to deal with the non-smoothness of $E$. In [ACO06] the non-differentiability is dealt with directly by considering the upper limit of the difference quotient. We take a slightly different route and use a *good cuboid width* near the suspected breakdown of the estimate. This adds variety and is somewhat quicker at the expense of at most a factor in the constants.

**Lemma 17.** *There is a universal constant $C_{L17} \geq 1$ permitting the following estimate. Given top and bottom magnetization as functions*

$$m_3^\mathrm{T}, m_3^\mathrm{B} : (-l,l)^2 \to [-1,1],$$

*and $(-l,l)^2$-periodic cumulated fields $H_\mathrm{T}'$, $H_\mathrm{B}' : (-l,l)^2 \to \mathbb{R}^2$ at top and bottom with*

$$\begin{aligned} -\nabla' \cdot H_\mathrm{T}' &= m_3^\mathrm{T} \text{ in } (-l,l)^2, \\ -\nabla' \cdot H_\mathrm{B}' &= m_3^\mathrm{B} \text{ in } (-l,l)^2, \end{aligned}$$

*let $m_3 : (-l,l)^2 \times (0, l_{x_3}) \to \{+1,-1\}$ and $h' : (-l,l)^2 \times (0,l_{x_3})$ be energy-minimizing among all $(-l,l)^2$-periodic configurations on $(-l,l)^2 \times (0, l_{x_3})$ satisfying*

$$\begin{aligned} \partial_3 m_3 + \cdot\nabla' \cdot h' &= 0 \text{ distributionally in } (-l_{x'}, l_{x'})^2 \times (0, l_{x_3}), \\ m_3 &\rightharpoonup m_3^\mathrm{B} \text{ weakly as } x_3 \to 0, \\ m_3 &\rightharpoonup m_3^\mathrm{T} \text{ weakly as } x_3 \to l_{x_3}, \\ \int_0^{l_{x_3}} h' dx_3 &= H_\mathrm{T}' - H_\mathrm{B}' \text{ in } (-l,l)^2. \end{aligned}$$

*Assume that there is $C_{H'} \geq 1$ such that*

$$\begin{aligned} \sup|H'_{\rm T}|, \sup|H'_{\rm B}| &\leq C_{H'} l_{x_3}^{2/3}, \\ l &\geq C_{L17} C_{H'} l_{x_3}^{2/3}. \end{aligned}$$

*Then for any $l_{x'} \geq C_{L17} C_{H'} l_{x_3}^{2/3}$ and any $x' \in (-l,l)^2$ we have*

$$\tilde{E}(l_{x'},x') = 2\int_{(x'+[-l_{x'},l_{x'})^2)\times(0,l_{x_3})} |\nabla' m_3| dx + \int_{(x'+(-l_{x'},l_{x'})^2)\times(0,l_{x_3})} |h' - \frac{1}{l_{x_3}}(H'_{\rm T} - H'_{\rm B})|^2 dx \leq C_{L17} l_{x_3}^{1/3} l_{x'}^2.$$

*If $H'_{\rm T} - H'_{\rm B}$ is curl-free then*

$$E(l_{x'},x') = 2\int_{(x'+[-l_{x'},l_{x'})^2)\times(0,l_{x_3})} |\nabla' m_3| dx + \int_{(x'+(-l_{x'},l_{x'}r)^2)\times(0,l_{x_3})} |h'|^2 dx \leq (C_{L17} + 8C_{H'}) l_{x_3}^{1/3} l_{x'}^2.$$

*Proof.* By translation we only need to be concerned with domains $(-l_{x'},l_{x'})^2 \times (0,l_{x_3})$ and can write $\tilde{E}(l_{x'}) := \tilde{E}(l_{x'},0)$. Let us assume

$$C_{\rm L17} \geq 8C_{\rm P5}.$$

An initial application of Proposition 5 on $(-l,l)^2 \times (0,l_{x_3})$ with $f(x',x_3) = \frac{1}{l_{x_3}}\nu' \cdot (H'_{\rm T} - H'_{\rm B})$ yields

$$\tilde{E}(l) \leq C_{\rm P5} l_{x_3}^{1/3} l^2.$$

We consider the energy on $(-2^{-i}l, 2^{-i}l) \times (0,l_{x_3})$. For $i=0$ the energy is bounded as desired. Assume that there is some smallest $i_*$ such that

$$\tilde{E}(2^{-i}l) \leq 8C_{\rm P5} l_{x_3}^{1/3} 2^{-2i} l^2 \quad \text{for } 0 \leq i \leq i_* \tag{94}$$

and

$$\tilde{E}(2^{-i_*-1}l) > 8C_{\rm P5} l_{x_3}^{1/3} 2^{-2i_*-2} l^2. \tag{95}$$

We now bound $2^{-i_*}l$. By Fubini's theorem and because when computing a mean, not every value can be above average, there exists a horizontal length $l_* \in (2^{-(i_*+1)}l, 2^{-i_*}l)$ such that

$$\begin{aligned} &\int_{\partial(-l_*,l_*)^2\times(0,l_{x_3})} |\nu' \cdot (h' - \frac{1}{l_{x_3}}(H'_{\rm T} - H'_{\rm B}))|^2 dx \\ \leq\ & 2^{i_*+1} l^{-1} \int_{(-2^{-i_*}l, 2^{-i_*}l)^2\times(0,l_{x_3})} |\nu' \cdot (h' - \frac{1}{l_{x_3}}(H'_{\rm T} - H'_{\rm B}))|^2 dx \\ \leq\ & 2^{i_*+1} l^{-1} \tilde{E}(2^{-i_*}l). \end{aligned} \tag{96}$$

Eyeing (82) we recognize that

$$\begin{aligned} \left(\int_{\partial(-l_*,l_*)^2\times(0,l_{x_3})} |\nu' \cdot (h' - \frac{1}{l_{x_3}}(H'_{\rm T} - H'_{\rm B}))|^2 dx\right)^{1/3} &\overset{(96)}{\leq} \left(2^{i_*+1} l^{-1} \tilde{E}(2^{-i_*}l)\right)^{1/3} \\ &\overset{(94)}{\leq} \left(2^{i_*+1} l^{-1} 8C_{\rm P5} l_{x_3}^{1/3} (2^{-i_*}l)^2\right)^{1/3} \\ &= 2C_{\rm P5}^{1/3} l_{x_3}^{1/9} 2^{-i_*/3+1/3} l^{1/3} \\ &\leq 2C_{\rm P5}^{1/3} l_{x_3}^{1/9} 2^{2/3} l_*^{1/3} \\ &\leq 2^{-3} l_{x_3}^{-1/3} l_* \end{aligned}$$

for $l_* \geq 2^7 C_{\mathrm{P5}}^{1/2} l_{x_3}^{2/3}$. By definition, the restriction of $H'_{\mathrm{T}}$ is compatible with the boundary values $\nu' \cdot h'$. Provided $l_* \geq \frac{16}{3} C_{H'}$ this allows us to plug (96) into the bound of Proposition 5 for $l_*$ and we see in combination with (94) and (95) that

$$\begin{aligned}
&2C_{\mathrm{P5}} l_{x_3}^{1/3} (2^{-i_*} l)^2 \\
&\quad = \quad 8C_{\mathrm{P5}} l_{x_3}^{1/3} 2^{-2i_*-2} l^2 \\
&\quad \overset{(95)}{\leq} \quad \tilde{E}(2^{-i_*-1} l) \\
&\quad \leq \quad \tilde{E}(l_*) \\
&\quad \overset{\text{Prop. 5}}{\leq} \quad C_{\mathrm{P5}} l_{x_3}^{1/3} \Bigg( \bigg( \int_{\partial(-l_*,l_*)^2 \times (0,l_{x_3})} |\nu' \cdot (h' - \frac{1}{l_{x_3}} (H'_{\mathrm{T}} - H'_{\mathrm{B}}))|^2 dx \bigg)^{4/3} \\
&\qquad\qquad + C_{H'} l_{x_3}^{1/3} \int_{\partial(-l_*,l_*)^2 \times (0,l_{x_3})} |\nu' \cdot (h' - \frac{1}{l_{x_3}} (H'_{\mathrm{T}} - H'_{\mathrm{B}}))|^2 dx + l_*^2 \Bigg) \\
&\quad \overset{(96)}{\leq} \quad C_{\mathrm{P5}} l_{x_3}^{1/3} \Big( \big( 2^{i_*+1} l^{-1} E(2^{-i_*} l) \big)^{4/3} + C_{H'} l_{x_3}^{1/3} 2^{i_*+1} l^{-1} E(2^{-i_*} l) + (2^{-i_*} l)^2 \Big) \\
&\quad \overset{(94)}{\leq} \quad C_{\mathrm{P5}} l_{x_3}^{1/3} \Big( \big( 2^{i_*+1} l^{-1} 8C_{\mathrm{P5}} l_{x_3}^{1/3} 2^{-2i_*} l^2 \big)^{4/3} + C_{H'} l_{x_3}^{1/3} 2^{i_*+1} l^{-1} 8C_{\mathrm{P5}} l_{x_3}^{1/3} 2^{-2i_*} l^2 + (2^{-i_*} l)^2 \Big) \\
&\quad = \quad C_{\mathrm{P5}}^{7/3} 2^4 l_{x_3}^{7/9} 2^{-4i_*/3+4/3} l^{4/3} + 8C_{\mathrm{P5}}^2 C_{H'} l_{x_3} 2^{-i_*+1} l + C_{\mathrm{P5}} l_{x_3}^{1/3} (2^{-i_*} l)^2.
\end{aligned}$$

Absorbing the last term into the left hand side we obtain

$$(2^{-i_*} l)^2 \leq C_{\mathrm{P5}}^{4/3} l_{x_3}^{4/9} 2^{-4i_*/3+16/3} l^{4/3} + C_{\mathrm{P5}} C_{H'} l_{x_3}^{2/3} 2^{-i_*+4} l$$

and somewhat lazily conclude

$$2^{-i_*} l \leq \max\{2^{19/2} C_{\mathrm{P5}}^2 l_{x_3}^{2/3}, 2^5 C_{\mathrm{P5}} C_{H'} l_{x_3}^{2/3}\}.$$

This is as desired when we choose

$$C_{\mathrm{L17}} := 4 \max\{2^{19/2} C_{\mathrm{P5}}^2, 2^5 C_{\mathrm{P5}}, \frac{16}{3}, 2^7 C_{\mathrm{P5}}^{1/2}\}$$

and recall $C_{H'} \geq 1$. The factor 4 is to extend the estimate from $2^{-i} l$ to arbitrary $l_{x'}$. The second estimate is analogous to the second statement in Proposition 5. It is not useful as long as we need very precise bounds but allows a nicer formulation in the final theorem. □

## 8.3 Decay of the cumulated field and local energy bounds

We now use the $x'$-local estimate of Lemma 17 to obtain information on $H'$ that allows us to consider cuboid subdomains with smaller $x_3$-extension. The technique and result of the next lemma resembles [Con00, Proposition 2.11] and the preparatory lemmas. We prefer to postpone specializing the boundary conditions for one more step, though. This is the point where we choose the $C_{H'}$ that we carefully tracked throughout Section 8.2.

**Lemma 18.** *Given a constant $C_0$ there is lower bound $C(C_0)$ depending only on $C_0$ such that for any $C_{H'} \geq C(C_0)$ the following estimate is valid: Let*

$$l_{x_3} \leq L_{x_3}$$

*and*

$$l \geq C_{H'} L_{x_3}^{2/3}$$

*and let* $(m_3, h')$ *be of minimal energy among* $(-l,l)^2$*-periodic configurations on* $(-l,l)^2 \times (0, l_{x_3})$ *satisfying*

$$\begin{aligned} \partial_3 m_3 + \nabla' \cdot h' &= 0 \text{ distributionally in } (-l,l)^2 \times (0, l_{x_3}), \\ m_3 &\rightharpoonup m_3^{\mathrm{B}} \text{ weakly as } x_3 \to 0, \\ m_3 &\rightharpoonup m_3^{\mathrm{T}} \text{ weakly as } x_3 \to l_{x_3}, \end{aligned}$$

*where the top magnetization and bottom magnetization are functions*

$$m_3^{\mathrm{T}}, m_3^{\mathrm{B}} : (-l,l)^2 \to [-1,1],$$

*such that the* $(-l,l)^2$*-periodic curl-free fields* $H'_{\mathrm{T}}$, $H'_{\mathrm{B}}$ *satisfy*

$$\begin{aligned} -\nabla' \cdot H'_{\mathrm{T}} &= m_3^{\mathrm{T}} \text{ in } (-l,l)^2, \\ -\nabla' \cdot H'_{\mathrm{B}} &= m_3^{\mathrm{B}} \text{ in } (-l,l)^2, \\ \int_0^{l_{x_3}} h' dx_3 &= H'_{\mathrm{T}} - H'_{\mathrm{B}} \text{ in } (-l,l)^2. \end{aligned}$$

*Assume*

$$\sup |H'_{\mathrm{T}} - H'_{\mathrm{B}}| \leq C_{H'} L_{x_3}^{1/3} l_{x_3}^{1/3}$$

*and that for any* $x' \in (-l,l)^2$

$$\int_{B_{C_{H'} L_{x_3}^{2/3}}(x') \times (0, l_{x_3})} |h' - \frac{1}{l_{x_3}} (H'_{\mathrm{T}} - H'_{\mathrm{B}})|^2 dx \leq C_0 C_{H'}^2 l_{x_3}^{1/3} L_{x_3}^{4/3}. \tag{97}$$

*Then the strength of the cumulated field*

$$H'(x', x_3) = H'_{\mathrm{B}} + \int_0^{x_3} h'(x', \xi_3) d\xi_3$$

*is close to the cumulated field at the boundary in the sense that at* $x_3 = l_{x_3}/2$

$$|H'(x', l_{x_3}/2) - H'_{\mathrm{B}}(x')|, |H'(x', l_{x_3}/2) - H'_{\mathrm{T}}(x')| \leq C_{H'} l_{x_3}^{1/3} L_{x_3}^{1/3}$$

*and for arbitrary* $x \in (-l,l) \times (0, l_{x_3})$

$$|H'(x) - H'_{\mathrm{B}}(x')|, |H'(x) - H'_{\mathrm{T}}(x')| \leq 2 C_{H'} l_{x_3}^{1/3} L_{x_3}^{1/3}.$$

The two lengths $l_{x_3}$ and $L_{x_3}$ should be thought of as the length of the cuboid and (after translation) the distance to the sample boundary. The latter influences the typical domain size. When we use the lemma for estimates at the sample boundary $l_{x_3}$ and $L_{x_3}$ coincide, but they differ substantially when we apply the lemma in the interior of the sample. The fact that we do not distinguish between them in the previous section can be compensated by scaling the constant $C_{H'}$ to be used in Lemma 17 by $l_{x_3}^{-2/3} L_{x_3}^{2/3}$. We should expect a better behavior of the energy in the interior, but that would require comparison constructions taking into account the fact that the top and bottom boundary magnetizations are not arbitrary in the interior.

*Proof of Lemma 18.* Let

$$H'_{\text{lin}}(x', x_3) = H'_{\text{B}} + \frac{x_3}{l_{x_3}}(H'_{\text{T}}(x') - H'_{\text{B}}(x'))$$

and consider for some fixed $\gamma \in (0,1)$

$$\sup_{(-l,l)^2\times\{\gamma l_{x_3}\}} |H' - H'_{\text{lin}}|.$$

We may assume that the supremum is attained at $x' = 0$. By definition, $H' - H'_{\text{lin}}$ vanishes for $x_3 \in \{0, l_{x_3}\}$. Thus we can estimate

$$\begin{aligned}
|(H' - H'_{\text{lin}})(x', \gamma l_{x_3})| &= \left|\int_0^{\gamma l_{x_3}} h'(x', \xi_3) - \frac{1}{l_{x_3}}(H'_{\text{T}}(x') - H'_{\text{B}}(x'))d\xi_3\right| \\
&\le (\gamma l_{x_3})^{1/2}\left(\int_0^{\gamma l_{x_3}} \left|h'(x', \xi_3) - \frac{1}{l_{x_3}}(H'_{\text{T}}(x') - H'_{\text{B}}(x'))\right|^2 d\xi_3\right)^{1/2} \quad \text{and} \\
|(H' - H'_{\text{lin}})(x', \gamma l_{x_3})| &= \left|\int_{\gamma l_{x_3}}^{l_{x_3}} h'(x', \xi_3) - \frac{1}{l_{x_3}}(H'_{\text{T}}(x') - H'_{\text{B}}(x'))d\xi_3\right| \\
&\le ((1-\gamma) l_{x_3})^{1/2}\left(\int_{\gamma l_{x_3}}^{l_{x_3}} \left|h'(x', \xi_3) - \frac{1}{l_{x_3}}(H'_{\text{T}}(x') - H'_{\text{B}}(x'))\right|^2 d\xi_3\right)^{1/2}.
\end{aligned}$$

Combining the square of the two, we have

$$\begin{aligned}
\frac{1}{\gamma(1-\gamma)l_{x_3}}|(H' - H'_{\text{lin}})(x', \gamma l_{x_3})|^2 &= \left(\frac{1}{\gamma l_{x_3}} + \frac{1}{(1-\gamma)l_{x_3}}\right)|(H' - H'_{\text{lin}})(x', \gamma l_{x_3})|^2 \\
&\le \int_0^{l_{x_3}} \left|h'(x', \xi_3) - \frac{1}{l_{x_3}}(H'_{\text{T}}(x') - H'_{\text{B}}(x'))\right|^2 d\xi_3.
\end{aligned}$$

Integrating in $x'$ over a ball $B_\rho := B_\rho(0)$ yields

$$\int_{B_\rho} |(H' - H'_{\text{lin}})(x', \gamma l_{x_3})|^2 dx' \le \gamma(1-\gamma) l_{x_3} \int_{B_\rho\times(0,l_{x_3})} |h' - \frac{1}{l_{x_3}}(H'_{\text{T}} - H'_{\text{B}})|^2 dx'. \tag{98}$$

By $|\nabla' \cdot H'| = |m_3| = 1$ we have

$$\sup_{(-l,l)^2\times\{\gamma l_{x_3}\}} |\nabla' \cdot (H' - H'_{\text{lin}})| \le \sup_{x'} |m_3(x', \gamma l_{x_3}) - \gamma m_3(x', l_{x_3})| \le 1 + \gamma. \tag{99}$$

From, say, [ACO06, Lemma 3.6] we take the standard estimate

$$|\nabla' u|^2(0) \le C_1 \int_{B_1} |\nabla' u|^2 dx' + C_1 \sup_{B_1} |\Delta' u|^2$$

and rescale to

$$|\nabla' u|^2(0) \le C_1 \rho^{-2} \int_{B_\rho} |\nabla' u|^2 dx' + C_1 \rho^2 \sup_{B_\rho} |\Delta' u|^2.$$

Because we assumed $(h', m_3)$ to be minimal, we know that $h'$ is minimal for fixed $m_3$ and thus $h'$ is a gradient field for almost every $x_3$ and so is $H'$ because $H'_{\text{B}}(x')$ is curl-free. Thus we can write

$H' - H'_{\mathrm{lin}} = -\nabla' u$ and obtain with (98) and (99) and assumption (97)

$$\begin{aligned} |(H' - H'_{\mathrm{lin}})(0, \gamma l_{x_3})|^2 &\leq C_1 \rho^{-2} \int_{B_\rho \times \{\gamma l_{x_3}\}} |H' - H'_{\mathrm{lin}}|^2 dx' + C_1 \rho^2 \sup_{(-l,l)^2 \times \{\gamma l_{x_3}\}} |\nabla' \cdot (H' - H'_{\mathrm{lin}})|^2. \\ &\leq C_1 \gamma (1-\gamma) l_{x_3} \rho^{-2} \int_{B_\rho \times (0, l_{x_3})} |h' - \frac{1}{l_{x_3}}(H'_{\mathrm{T}} - H'_{\mathrm{B}})|^2 dx + C_1 \rho^2 (1+\gamma)^2 \\ &\overset{(97)}{\leq} C_1 \gamma (1-\gamma) l_{x_3}^{4/3} \rho^{-2} C_0 C_{H'}^2 L_{x_3}^{4/3} + C_1 \rho^2 (1+\gamma)^2, \end{aligned}$$

provided $\rho \leq C_{H'} L_{x_3}^{2/3}$. By requiring $C_{H'} \geq C_0^{1/2}$ we may optimize in $\rho$ to obtain

$$\begin{aligned} |(H' - H'_{\mathrm{lin}})(0, \gamma l_{x_3})|^2 &\leq C_1 C_0^{1/2} \gamma^{1/2} (1-\gamma)^{1/2} (1+\gamma) C_{H'} l_{x_3}^{2/3} L_{x_3}^{2/3} \\ &\leq C_1 C_0^{1/2} C_{H'} l_{x_3}^{2/3} L_{x_3}^{2/3}. \end{aligned}$$

Let us emphasize the scaling in $H'$ here: Keeping in mind that our goal is to estimate $H'$ in terms of $C_{H'} L_{x_3}^{1/3} l_{x_3}^{1/3}$, the bound is quadratic in $C_{H'}$ on the left hand side but only linear on the right.

We focus on $\gamma = \frac{1}{2}$. By *choosing* $C_{H'}$ to satisfy

$$\left( \frac{1}{2^{2/3}} - \frac{1}{2} \right) C_{H'} \geq C_1^{1/2} C_0^{1/4} C_{H'}^{1/2}$$

or equivalently

$$C_{H'} \geq 4(2^{1/3} - 1)^{-2} C_1 C_0^{1/2}$$

we can estimate

$$\begin{aligned} \sup_{(-l,l)^2 \times \{\frac{1}{2} l_{x_3}\}} |H' - H'_{\mathrm{B}}| &\leq \frac{1}{2} \sup_{(-l,l)^2} |H'_{\mathrm{T}} - H'_{\mathrm{B}}| + \sup_{(-l,l)^2 \times \{\frac{1}{2} l_{x_3}\}} |H' - H'_{\mathrm{lin}}| \\ &\leq 2^{-2/3} C_{H'} l_{x_3}^{1/3} L_{x_3}^{1/3} \end{aligned}$$

and similarly

$$\sup_{(-l,l)^2 \times \{\frac{1}{2} l_{x_3}\}} |H' - H'_{\mathrm{T}}| \leq 2^{-2/3} C_{H'} l_{x_3}^{1/3} L_{x_3}^{1/3}.$$

For arbitrary $\gamma$ our information is not as precise and we lose the exact scaling, but we still achieve

$$\sup_{(-l,l)^2 \times (0, l_{x_3})} |H' - H'_{\mathrm{B}}|, \sup_{(-l,l)^2 \times (0, l_{x_3})} |H' - H'_{\mathrm{T}}| \leq \left( \frac{1}{2} + \frac{1}{2^{2/3}} \right) C_{H'} l_{x_3}^{1/3} L_{x_3}^{1/3},$$

as claimed. □

Iterating Lemma 18 we obtain the two-dimensional equivalent of [Con00, Theorem 2.1].

**Theorem 5.** *There is a universal constant $C_{H'}$ such that the following holds. Let*

$$l \geq 4 C_{H'} l_{x_3,0}^{2/3}$$

*and let $(m_3, h')$ be of minimal energy among $(-l,l)^2$-periodic configurations on $(-l,l)^2 \times (0, l_{x_3,0})$ satisfying*

$$\begin{aligned} \partial_3 m_3 + \nabla' \cdot h' &= 0 \text{ distributionally in } (-l,l)^2 \times (0, l_{x_3,0}), \\ m_3 &\rightharpoonup 0 \text{ weakly as } x_3 \to 0, \\ m_3 &\rightharpoonup m_3^{\mathrm{T}} \text{ weakly as } x_3 \to l_{x_3}, \end{aligned}$$

*where the top magnetization is a function*

$$m_3^{\mathrm{T}} : (-l,l)^2 \to [-1,1],$$

*such that the* $(-l,l)^2$*-periodic field* $H'_{\mathrm{T}}(x') = \int_0^{l_{x_3}} h'(x',x_3)dx_3$ *satisfies*

$$-\nabla' \cdot H'_{\mathrm{T}} = m_3^{\mathrm{T}} \text{ in } (-l,l)^2$$

*and*

$$\sup |H'_{\mathrm{T}}| \le C_{H'} l_{x_3,0}^{2/3}.$$

*Then the strength of the cumulated field*

$$H'(x',x_3) = \int_0^{x_3} h'(x',\xi_3)d\xi_3$$

*decays as*

$$\sup_{(-l,l)^2\times\{l_{x_3}\}} |H'| \le C_{H'} l_{x_3}^{2/3} \qquad \text{for any } l_{x_3} < l_{x_3,0}. \tag{100}$$

*On any cuboid* $y + ((-l_{x'},l_{x'})^2 \times (0,l_{x_3})) \subseteq (-l,l)^2 \times (0,l_{x_3,1})$ *with* $l_{x'} \ge C_{L17} C_{H'} l_{x_3,1}^{2/3}$ *the energy is bounded by*

$$2\int_{(y'+[-l_{x'},l_{x'})^2)\times(y_3,y_3+l_{x_3})} |\nabla' m_3|dx + \int_{(y'+(-l_{x'},l_{x'})^2)\times(y_3,y_3+l_{x_3})} |h'(x) - \frac{1}{l_{x_3}}(H'(x',y_3+l_{x_3}) - H'(x',y_3))|^2 dx \le C_{L17} l_{x_3}^{1/3} l_{x'}^2,$$

*in particular, for cuboids at the sample boundary*

$$2\int_{(y'+[-l_{x'},l_{x'})^2)\times(0,l_{x_3})} |\nabla' m_3|dx + \int_{(y'+(-l_{x'},l_{x'})^2)\times(0,l_{x_3})} |h|^2 dx \le C l_{x_3}^{1/3} l_{x'}^2.$$

*Furthermore* $H' \in C^{0,1/3}((-l,l)\times(0,l_{x_3,1}))$ *with*

$$|H'(x) - H'(y)| \le C_{H'}(1 + l_{x_3,1}^{2/3})|x-y|^{1/3}.$$

*Proof.* We iterate Lemma 18 with $l_{x_3} := L_{x_3} := 2^{-k} l_{x_3,0}$ and $C_0 := C_{\mathrm{L17}}^3$ and $H'_B \equiv 0$ to obtain (100) for $l_{x_3}$ of this form and, after enlarging $C_{H'}$ by a factor of two, also on $l_{x_3} \in (2^{-k} l_{x_3,0}, 2^{-k-1} l_{x_3,0})$, i.e. all $l_{x_3} < l_{x_3,0}$ after the iteration. For the local energy bound, we use Lemma 18 with $C_{H'}$ replaced by $l_{x_3}^{-2/3} l_{x_3,1}^{2/3} C_{H'}$.

We now turn to the Hölder continuity. Iterating Lemma 18, this time with $L_{x_3} := l_{x_3,1}$, $l_{x_3} := 2^{-k} l_{x_3,1}$, $H'_B(x') := H'(x',(1-2^{-k})l_{x_3,1})$, and $H'_T(x') := H'(x',l_{x_3,1})$, we obtain

$$|H'(x',x_3) - H'(x',y_3)| \le C_{H'} l_{x_3,1}^{1/3} |x_3 - y_3|^{1/3}.$$

For the horizontal direction we employ the standard interior elliptic estimates after writing $H'$ as a gradient again. From

$$\sup_{x',y'\in B_{1/2}} \frac{|\nabla' u(x') - \nabla' u(y')|}{|x'-y'|^\alpha} \le C(\alpha)(\|\Delta' u\|_{L^{2/(1-\alpha)}(B_1)} + \|\nabla' u\|_{L^2(B_1)}),$$

valid for any $0 < \alpha < 1$, see e.g. [HL97, Theorems 3.1 and 3.13], we conclude

$$\sup_{x',y' \in B_{1/2}} \frac{|H'(x',x_3) - H'(y',x_3)|}{|x'-y'|^\alpha} \le C(\sup_{B_1} |\nabla' \cdot H'| + \sup_{B_1} |H'(x',x_3)|) \le C(1 + C_{H'} l_{x_3}^{2/3}).$$

Combined with the boundedness of $H'$ we have that $H' \in C^{0,1/3}$ with norm bounded by $C_{H'}(1 + l_{x_3,1}^{2/3})$ after replacing $C_{H'}$ with $CC_{H'}$. □

We remark that the theorem is applicable with $H'_T \equiv 0$ in order to bootstrap the argument starting from the full sample.

## 8.4 Blowup at the sample boundary

Using the local energy bound from Theorem 5 we can prove $L^1_{\text{loc}}$-compactness (for $m_3$) of blow-up sequences.

**Lemma 19.** *Let $m_3$, $h'$ be a minimizing $(-l,l)^2$-periodic configuration as in Theorem 5.*

*Consider $m_3$, $h'$ as periodically extended to $\mathbb{R}^2 \times (0, l_{x_3})$. Then any blow-up sequence*

$$\begin{aligned} m_3^{(j)}(x',x_3) &= m_3(\theta^j(x'-x'_0), \theta^{3j/2}x_3), \\ h'^{(j)}(x',x_3) &= \theta^{j/2}h'(\theta^j(x'-x'_0), \theta^{3j/2}x_3) \end{aligned}$$

*for some $\theta < 1$ has a subsequence such that*

$$\begin{aligned} m_3^{(j)} &\to m_3^* \text{ in } L^1_{\text{loc}} \text{ and a.e.}, \\ h'^{(j)} &\rightharpoonup h'^* \text{ weakly in } L^2_{\text{loc}}, \\ H'^{(j)} &\to H'^* \text{ in } C^0_{\text{loc}}, \\ H'^{(j)} &\rightharpoonup H'^* \text{ weakly in } H^1_{\text{loc}}, \end{aligned}$$

*in the sense that for any given compact domain after dropping finitely many items the restriction of the functions in the sequence converges.*

*Proof.* We fix some cuboid $Q := Q(a) := (-a,a)^2 \times (0, (C_{\text{L17}} C_{H'})^{-3/2} a^{3/2})$ and show convergence on $Q$ for a subsequence. Taking a diagonal subsequence for a series of cuboids, say $Q(2^k)$, yields the full result on $\mathbb{R}^2 \times (0,\infty)$.

We have three uniform bounds for the sequence to work with. Trivially

$$\|m_3^{(j)}\|_{L^\infty} = 1$$

and by the energy bound of Theorem 5

$$\begin{aligned} \int_Q |\nabla' m_3^{(j)}| dx &\le C(a), \\ \int_Q |h'^{(j)}|^2 dx &\le C(a). \end{aligned}$$

By the $L^\infty$-bound for $m_3^{(j)}$ we know that for a subsequence

$$m_3^{(j)} \rightharpoonup m_3^* \text{ weakly-* in } L^\infty(Q).$$

We want to use the other two bounds to see that the convergence is in fact strong in $L^1$. We thus want to show that

$$\int_{(-a+\alpha,a-\alpha)^2\times(0,(C_{\mathrm{L}17}C_{H'})^{-3/2}a^{3/2})} |m_3^{(j)}(x'+y',x_3)-m_3^{(j)}(x',x_3)|dx \leq C|y'| \quad \text{for } |y'|\leq\alpha, \tag{101}$$

$$\left(\int_{(-a+\alpha,a-\alpha)^2\times(\alpha^{3/2},(C_{\mathrm{L}17}C_{H'})^{-3/2}a^{3/2}-\alpha^{3/2})} |m_3^{(j)}(x',x_3+y_3)-m^{(j)}(x',x_3)|^2dx\right)^{1/2} \leq C|y_3|^{1/3} \quad \text{for } |y_3|\leq\alpha^{3/2}, \tag{102}$$

and then apply the compactness criterion of M. Riesz (see e.g. [Ada75, Theorem 2.21]).

The first inequality (101) is a direct consequence of the $BV$-bound, i.e.

$$\int_{(-a+\alpha,a-\alpha)^2\times(0,(C_{\mathrm{L}17}C_{H'})^{-3/2}a^{3/2})} |m_3^{(j)}(x'+y',x_3)-m_3^{(j)}(x',x_3)|dx \leq |y'|\int_Q |\nabla' m_3^{(j)}|dx \leq |y'|C.$$

For the inequality (102) we use a compensated-compactness argument leveraging the weak control over $\partial_3 m_3$ with the stronger control on $\nabla' m_3$. Let $(f)_\alpha$ denote the convolution (w.r.t. $x'$) of any function $f$ with a scaled standard mollifier $\varphi_\alpha(x')=\alpha^{-2}\varphi_1(\alpha^{-1}x')$, $\varphi_1\in C_c^\infty(B_1(0))$, $\varphi_1\geq 0$ with mass 1. From the compatibility equation for $m_3^{(j)}$, $h'^{(j)}$ we deduce

$$\partial_3(m_3^{(j)})_\alpha+\nabla'\cdot(h'^{(j)})_\alpha=0$$

in $(-a+\alpha,a-\alpha)^2\times(0,(C_{\mathrm{L}17}C_{H'})^{-3/2}a^{3/2})$.

In particular, by the standard trick of estimating the divergence of the convolution by differentiation of the mollifier

$$\begin{aligned}\|\partial_3(m_3^{(j)})_\alpha\|_{L^2((-a+\alpha,a-\alpha)^2\times(0,(C_{\mathrm{L}17}C_{H'})^{-3/2}a^{3/2}))} &= \|\nabla'\cdot(h'^{(j)})_\alpha\|_{L^2((-a+\alpha,a-\alpha)^2\times(0,(C_{\mathrm{L}17}C_{H'})^{-3/2}a^{3/2}))}\\ &\leq C\alpha^{-1}\|h'^{(j)}\|_{L^2(Q)}\\ &\leq C\alpha^{-1}.\end{aligned}$$

Integrating over an $x_3$-interval of length $\tau$ we obtain

$$\|(m_3^{(j)})_\alpha(x',x_3+\tau)-(m_3^{(j)})_\alpha(x',x_3)\|_{L^2((-a+\alpha,a-\alpha)^2\times(\tau,(C_{\mathrm{L}17}C_{H'})^{-3/2}a^{3/2}-\tau))}\leq C|\tau|\alpha^{-1}.$$

Finally, we bound the difference of the convolution to $m_3^{(j)}$ with the help of estimate (101) for the $x'$-modulus of continuity. This yields

$$\begin{aligned}&\|m_3^{(j)}(x',x_3+\tau)-m_3^{(j)}(x',x_3)\|_{L^2((-a+\alpha,a-\alpha)^2\times(\tau,(C_{\mathrm{L}17}C_{H'})^{-3/2}a^{3/2}-\tau))}\\ &\leq \|(m_3^{(j)})_\alpha(x',x_3+\tau)-(m_3^{(j)})_\alpha(x',x_3)\|_{L^2((-a+\alpha,a-\alpha)^2\times(\tau,(C_{\mathrm{L}17}C_{H'})^{-3/2}a^{3/2}-\tau))}\\ &\quad +2\|(m_3^{(j)})_\alpha-m_3^{(j)}\|_{L^2((-a+\alpha,a-\alpha)^2\times(0,(C_{\mathrm{L}17}C_{H'})^{-3/2}a^{3/2}))}\\ &\leq C|\tau|\alpha^{-1}+2\sup_{|y'|\leq\alpha}\|m_3^{(j)}(x'+y',x_3)-m_3^{(j)}(x',x_3)\|_{L^2((-a+\alpha,a-\alpha)^2\times(0,(C_{\mathrm{L}17}C_{H'})^{-3/2}a^{3/2}))}\\ &\leq C|\tau|\alpha^{-1}+2\sup_{|y'|\leq\alpha}\|m_3^{(j)}(x'+y',x_3)-m_3^{(j)}(x',x_3)\|_{L^2((-a+\alpha,a-\alpha)^2\times(0,(C_{\mathrm{L}17}C_{H'})^{-3/2}a^{3/2}))}\\ &\leq C|\tau|\alpha^{-1}+2^{3/2}\sup_{|y'|\leq\alpha}\|m_3^{(j)}(x'+y',x_3)-m_3^{(j)}(x',x_3)\|^{1/2}_{L^1((-a+\alpha,a-\alpha)^2\times(0,(C_{\mathrm{L}17}C_{H'})^{-3/2}a^{3/2}))}\\ &\overset{(101)}{\leq} C|\tau|\alpha^{-1}+C\alpha^{1/2}.\end{aligned}$$

Choosing the optimal $\alpha = |\tau|^{2/3}$ we obtain (102). The uniform $L^\infty$-bound implies that the $L^1$-norm in a boundary layer vanishes uniformly as the width converges to 0. This combined with (101) and (102) allows us to conclude with the theorem of M. Riesz that the sequence is precompact in $L^1$. Thus a subsequence $m_3^{(j)}$ converges in $L^1$ (and any $L^p$ with $p < \infty$) and almost everywhere to $m_3^*$ and in particular $|m_3^*| = 1$ a.e.

From interior elliptic regularity theory (writing $H'$ as a gradient) we know that

$$\begin{aligned}
&\int_Q |\nabla' H'^{(j)}|^2 dx \\
&\lesssim \int_{(-a-1,a+1)^2\times(0,(C_{\mathrm{L17}}C_{H'})^{-3/2}a^{3/2})} |H'^{(j)}|^2 dx + \int_{(-a-1,a+1)^2\times(0,(C_{\mathrm{L17}}C_{H'})^{-3/2}a^{3/2})} |m_3^{(j)}|^2 dx \\
&\leq C(a).
\end{aligned}$$

Thus $H'^{(j)}$ is bounded in $W^{1,2}(Q)$ and so for a subsequence

$$H'^{(j)} \rightharpoonup H'^* \text{ weakly in } W^{1,2}(Q).$$

Finally, boundedness in $C^{0,\delta}$ implies compactness in $C^0$ by Arzela-Ascoli's theorem and, taking a further subsequence if necessary, we have

$$H'^{(j)} \to H'^* \text{ strongly in } C^0(Q),$$

so we have established convergence for $Q$. As indicated in the beginning, we obtain the full result by taking a diagonal sequence over some exhaustion of the half space. □

# 9 Appendix

## 9.1 Stray field

For the reader's convenience we collect some facts related to the treatment of the stray field and our notation involving the inverted divergence in this appendix.

As the magnetization induces a stray field $h$, the conceptually simplest way to include its contribution to the energy is to explicitly include the squared $L^2$-norm

$$\int_{\mathbb{R}^3} |h|^2 dx$$

in the energy. The stray field $h$ satisfies Maxwell's equations (greatly reduced in the magnetostatic case to)

$$\nabla \cdot (h + m) = 0 \text{ and } \quad \nabla \times h = 0, \tag{103}$$

both understood in the sense of distributions on $\mathbb{R}^3$. For notes on the derivation, see e.g. [DKMO05]. Being curl-free, $h$ is a gradient field and, in fact, the Helmholtz projection of $-m$ onto the space of gradient fields. One way to compute $h$ is setting $h = -\nabla u$ where

$$\begin{aligned}
\Delta u &= \nabla \cdot m \quad \text{in } \Omega, \\
\left[\tfrac{\partial u}{\partial \nu}\right] &= m \cdot \nu \quad \text{on } \partial\Omega, \\
\Delta u &= 0 \qquad \text{outside } \Omega.
\end{aligned} \tag{104}$$

We can similarly define $h$ for periodic domains, then (104) reduces to the first equation $\Delta u = \nabla \cdot m$.

An alternative approach to the stray-field energy is to include $h$ in the minimization in order to make the problem more local. Observe that the $L^2$-norm of $(-l,l)^2$-periodic $h : \mathbb{R}^3 \to \mathbb{R}^3$ defined by (103) can be rewritten in terms of the minimization problem

$$\int_{(-l,l)^2\times\mathbb{R}} |h|^2 dx = \min\Big\{ \int_{(-l,l)^2\times\mathbb{R}} |\tilde{h}|^2 dx \Big| \, \tilde{h} : \mathbb{R}^3 \to \mathbb{R}^3 \text{ is } (-l,l)^2\text{-periodic in } x', \quad \nabla\cdot(\tilde{h}+m) = 0 \text{ distributionally in } \mathbb{R}^3 \Big\}, \tag{105}$$

and the second equation in (103) is just the Euler-Lagrange equation for the minimization. Hence, setting

$$e_{Q,d,t,l}(m,h) := \frac{1}{4l^2}\left( d^2 \int_\Omega |\nabla m|^2 dx + Q\int_\Omega |m'|^2 dx + \int_{\mathbb{R}^3} |h|^2 dx \right)$$

we have

$$e(Q,d,t,l) = \min\Big\{ e_{Q,d,t,l}(m,h) \Big| \, m,h : \mathbb{R}^3 \to \mathbb{R}^3 \text{ are } (-l,l)^2\text{-periodic in } x', \quad |m|^2 = \begin{cases} 1 & \text{for } x_3 \in (-t,t), \\ 0 & \text{otherwise,} \end{cases} \quad \nabla\cdot(h+m) = 0 \text{ distributionally in } \mathbb{R}^3 \Big\}.$$

There is a third way to think about $h$ that we want to illustrate with the stray-field term in the reduced energy concerning $m_3 : \mathbb{R}^3 \to \{-1,1\}$ and $h' : \mathbb{R}^3 \to \mathbb{R}^2$, both $(-l,l)^2$-periodic in $x_3$ and satisfying $m_3^2 = 1$ if $x_3 \in (-1,1)$ and $m_3^2 = 0$ otherwise and

$$\nabla' h + \partial_3 m_3 = 0.$$

We are tempted to invert the operator $\nabla'$ in the above equation. Indeed we define for any distribution $f$ (with zero slicewise average)

$$\int_{(-l,l)^2\times\mathbb{R}} \Big||\nabla'|^{-1} f\Big|^2 dx = \min\Big\{ \int_{(-l,l)^2\times\mathbb{R}} |\tilde{h}'|^2 dx \Big| \, \tilde{h} : \mathbb{R}^3 \to \mathbb{R}^3 \text{ is } (-l,l)^2\text{-periodic in } x', \quad \nabla'\cdot\tilde{h}' = f \text{ distributionally in } \mathbb{R}^3 \Big\}$$

and can thus write in the spirit of (105)

$$\int_{(-l,l)^2\times\mathbb{R}} |h'|^2 dx = \int_{(-l,l)^2\times\mathbb{R}} \Big||\nabla'|^{-1}\partial_3 m\Big|^2 dx.$$

Another way to look at $|\nabla'|^{-1}$ is by taking Fourier series in $x'$-direction. With

$$\mathcal{F}'(\zeta)(n') = \frac{1}{2l}\int_{(-l,l)^2} \exp\left(-\pi i n' \cdot \frac{x'}{l}\right) \zeta(x') dx'$$

and

$$\int_{(-l,l)^2\times\mathbb{R}} \big||\nabla'|^{-1} f\big|^2 dx = \int_{\mathbb{R}} \sum_{n'\in\mathbb{Z}^2} \frac{l^2}{\pi^2 |n'|^2} |(\mathcal{F}' f)(n')|^2 dx_3$$

we can rewrite the energy as

$$\int_{(-l,l)^2\times\mathbb{R}} |h'|^2 dx = \int_{(-l,l)^2\times\mathbb{R}} \left||\nabla'|^{-1}\partial_3 m_3\right|^2 dx = \int_{\mathbb{R}} \sum_{n'\in\mathbb{Z}^2} \frac{l^2}{\pi^2|n'|^2} |(\mathcal{F}'(\partial_3 m_3))(n')|^2 dx_3.$$

This also aligns well to the method of defining the energy via (104), when we plug in the usual Fourier-series solution formula for Poisson's equation on periodic domains.

Let us briefly look at the rôle of this inverse norm as a dual of the $H^1$-seminorm making a brief appearance in the proof of the interpolation inequality Lemma 4. Fix two $(-l,l)^2$-periodic functions $f, g$ with average 0, thought of as smooth, and let $u$ be a solution to $\Delta u = g$. Then by the divergence theorem and the Cauchy-Schwarz inequality, the duality estimate is but a simple calculation

$$\begin{aligned}
\int_{(-l,l)^2} fg\, dx' &= -\int_{(-l,l)^2} f\Delta u\, dx' \\
&= \int_{(-l,l)^2} \nabla f\cdot\nabla u\, dx' \\
&\le \left(\int_{(-l,l)^2} |\nabla f|^2 dx'\right)^{1/2} \left(\int_{(-l,l)^2} |\nabla u|^2 dx'\right)^{1/2} \\
&= \left(\int_{(-l,l)^2} |\nabla f|^2 dx'\right)^{1/2} \left(\int_{(-l,l)^2} \left||\nabla|^{-1} g\right|^2 dx'\right)^{1/2}. \qquad (106)
\end{aligned}$$

## 9.2 A Poincaré inequality

There are so many variants of the Poincaré inequality in the literature that it seems hard to find one matching our specific needs. We include the following for convenience.

**Lemma 20.** *Let $Q_1 := (0,l)^n$. Any function*

$$f : (0,l)^n \to \mathbb{R}$$

*with derivatives in $L^p$ such that the mean over some subcuboid $Q_0 := \prod_{i=1}^n (a_i, a_i + b_i)$ vanishes satisfies the Poincaré estimate*

$$\int_{Q_1} |f|^p dx \le C(n,p)\frac{|Q_1'|}{|Q_0'|} l^p \int_{Q_1} |\nabla f|^p dx$$

*where the $Q_i'$ are the projections of the $Q_i$ into one axis-parallel $n-1$-dimensional subspace. The constant $C(n,p)$ only depends on the dimension $n$ and exponent $p$ and not on $f$, $l$, $a_i$, or, importantly, $b_i$.*

Considering a smooth function vanishing on $(0,1)^n$ and constant 1 on $(0,l)^n \setminus (0,2)^n$ we see that the scaling is optimal for $p = 1$. For larger $p$ it is just good enough for our purposes.

*Proof.* We start in one dimension and so consider

$$f : (0,l) \to \mathbb{R}$$

with

$$\frac{1}{b}\int_a^{a+b} f dx = 0$$

where $(a, a+b) \subseteq (0,l)$. Averaging

$$f(x) = f(t) + \int_t^x f'(\tau)d\tau.$$

we obtain

$$\begin{aligned}
f(x) &= \frac{1}{b}\int_{(a,a+b)} \left( f(t) + \int_t^x f'(\tau)d\tau \right) dt \\
&= \frac{1}{b}\int_{(a,a+b)} \int_t^x f'(\tau)d\tau dt \\
&= \frac{1}{b}\left( \int_{(a,\min\{a+b,x\})} \int_t^x f'(\tau)d\tau dt - \int_{(\max\{a,x\},a+b)} \int_x^t f'(\tau)d\tau dt \right) \\
&= \int_{(a,x)} \min\{\frac{t-a}{b}, 1\} f'(t)dt - \int_{(x,a+b)} \min\{\frac{a+b-t}{b}, 1\} f'(t)dt.
\end{aligned}$$

Note that for $t_1 < t_2$ we use the usual conventions $\int_{t_2}^{t_1} = -\int_{t_1}^{t_2}$ but $(t_2, t_1) = \emptyset$ and so $\int_{(t_2,t_1)} = 0$. We plug this expression into the integral for the $L^p$-norm and get

$$\begin{aligned}
\int_0^l |f|^p dx &= \int_0^l \left| \int_{(a,x)} \min\{\frac{t-a}{b}, 1\} f'(t)dt - \int_{(x,a+b)} \min\{\frac{a+b-t}{b}, 1\} f'(t)dt \right|^p dx \\
&\leq \int_0^l \left| \int_{(a,x)} \min\{\frac{t-a}{b}, 1\} |f'(t)|dt + \int_{(x,a+b)} \min\{\frac{a+b-t}{b}, 1\} |f'(t)|dt \right|^p dx \\
&\leq \int_0^l \left| \int_0^l |f'(t)|dt \right|^p dx \\
&\leq l^p \int_0^l |f'(t)|^p dt.
\end{aligned}$$

This is the desired estimate in one dimension.

For higher dimensions we assume w.l.o.g. that $b_n$ is minimal among the $b_i$ and let $Q_1 := (0,l)^n$, $Q_1' := (0,l)^{n-1}$, $Q_2 := \prod_{i=1}^n (a_i, a_i + b_i)$ and $Q_2' := \prod_{i=1}^{n-1}(a_i, a_i + b_i)$ and $f_M := |M|^{-1} \int_M f dx^{(\prime)}$. We note that

$$f_{Q_1'(x_n)} = \left(f_{Q_1'(x_n)} - f\right)_{Q_0'(x_n)} + (f)_{Q_1'(x_n)}$$

and thus with Jensen's inequality and its elementary variant $(a+b)^p \leq 2^{p-1}(a^p + b^p)$

$$|f_{Q_1'(x_n)}|^p \leq 2^{p-1}\left(|f_{Q_1'(x_n)} - f|^p\right)_{Q_0'(x_n)} + 2^{p-1}|(f)_{Q_0'(x_n)}|^p. \tag{107}$$

Applying the one-dimensional estimate to

$$g(x_n) := f_{Q_0'(x_n)}$$

we have that

$$\int_{(0,l)} |f_{Q_0'(x_n)}|^p dx_n \leq l^p \int_{(0,l)} |\partial_n f_{Q_0'(x_n)}|^p dx_n \leq l^{p+1} (|\nabla f|^p)_{Q_0' \times (0,l)}.$$

Integrating (107) we obtain

$$\begin{aligned}
|f_{Q_1}|^p &\le l^{-1}\int_{(0,l)} |f_{Q_1'(x_n)}|^p dx_n \\
&\le l^{-1}2^{p-1}\int_{(0,l)} \left(|f_{Q_1'(x_n)} - f|^p\right)_{Q_0'(x_n)} dx_n + l^{-1}2^{p-1}\int_{(0,l)} |(f)_{Q_0'(x_n)}|^p dx_n \\
&\le l^{-1}2^{p-1}\int_{(0,l)} \left(|f_{Q_1'(x_n)} - f|^p\right)_{Q_0'(x_n)} dx_n + 2^{p-1}l^p(|\nabla f|^p)_{Q_0'\times(0,l)} \\
&\le l^{-1}2^{p-1}\int_{(0,l)} |Q_0'|^{-1}\int_{Q_1'(x_n)} |f_{Q_1'(x_n)} - f|^p dx' dx_n + 2^{p-1}l^p(|\nabla f|^p)_{Q_0'\times(0,l)}.
\end{aligned}$$

Appealing to the usual Poincaré inequality

$$\int_{(0,l)^d} \left|f - (f)_{(0,l)^d}\right|^p dx \le C_{p,d}\, l^p \int_{(0,l)^d} |\nabla f|^p dx$$

we estimate

$$\begin{aligned}
|f_{Q_1}|^p &\le l^{-1}2^{p-1}\int_{(0,l)} |Q_0'|^{-1}\int_{Q_1'(x_n)} |f_{Q_1'(x_n)} - f|^p dx' dx_n + 2^{p-1}l^p(|\nabla f|^p)_{Q_0'\times(0,l)} \\
&\le C_{n-1,p}l^{-1+p}2^{p-1}|Q_0'|^{-1}\int_{Q_1} |\nabla f|^p dx + 2^{p-1}l^p(|\nabla f|^p)_{Q_0'\times(0,l)} \\
&\le (C_{n-1,p}+1)l^{-1+p}2^{p-1}|Q_0'|^{-1}\int_{Q_1} |\nabla f|^p dx.
\end{aligned}$$

With this estimate and the regular Poincaré inequality for $d = n$ we conclude

$$\begin{aligned}
\int_{Q_1} |f|^p dx &\le 2^{p-1}\int_{Q_1} |f - f_{Q_1}|^p dx + 2^{p-1}|Q_1|\,|f_{Q_1}|^p \\
&\le 2^{p-1}C_{p,n}l^p\int_{Q_1} |\nabla f|^p dx + 2^{2p-2}(C_{n-1,p}+1)|Q_1'|l^p|Q_0'|^{-1}\int_{Q_1} |\nabla f|^p dx \\
&\le C\frac{|Q_1'|}{|Q_0'|}l^p\int_{Q_1} |\nabla f|^p dx.
\end{aligned}$$

This covers the case of arbitrary dimension. □

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